Membership Algebra as a Logical Framework for Equational Specification*

José Meseguer

SRI International, Menlo Park, CA 94025, USA
meseguer@csl.sri.com

A mi madre, Fuensanta Guaita de Meseguer,
con todo cariño en su 85 cumpleaños

Abstract. This paper proposes *membership equational logic*—a Horn logic in which the basic predicates are equations $t = t'$ and membership assertions $t : s$ stating that a term t belongs to a sort s—as a logical framework in which a very wide range of total and partial equational specification formalisms can be naturally represented. Key features of this logic include: simplicity, liberality and equational character; generality and expressiveness in supporting subsorts, overloading, errors and partiality; and efficient implementability in systems such as Maude. The paper presents the basic properties of the logic and its models, and discusses in detail how many total and partial equational specification formalisms, including order-sorted algebra and *partial* membership equational logic, can be represented in it, as well as the practical benefits in terms of tool reusability that this opens up for other languages, including CASL.

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1 Introduction

This work is an intermediate stage in a long-term quest for ever better equational specification formalisms. The fact that many of us have been at it for quite some time and that this has already produced a rich collection of formalisms should not be an excuse to lose nerve and give up the quest. We should press on.

In fact, the time may be ripe for a task of reflection and unification to try to pull together and understand in a more holistic way the best ideas that have been developed in the algebraic specification research program. This program, both in its core ideas and in its ever wider applications to other areas such as object-oriented programming, concurrency, software engineering, theorem proving, and logical frameworks, has yielded very powerful techniques and has shown sufficient maturity and promise as to make Joseph Goguen state that we may be entering "a golden era of algebraic specification." One may indeed wonder whether the next millennium might not in fact usher in such a marvelous event.

Whatever our degree of optimism may be about such predictions, it seems undeniable that strong winds, of a syncretic and ecumenical nature, are blowing around us and are stimulating much activity and excitement in projects such as the Common Algebraic Specification Language (CASL) [9]. There are good pragmatic reasons for this. It takes long years of hard work to develop well-finished and well-documented *systems* and *tools* that are essential to transfer our techniques and ideas to the world of engineering practice. An atomization of our energies into countless formalisms and experimental systems without a way to integrate them and to have good tool support for them would be a sad waste of efforts, and perhaps a lack of scientific responsibility.

All this makes the idea of a *universal language* or *logical framework* for equational specification[2] quite attractive. This paper is in fact a proposal for a specific logical framework, *membership equational logic*, in which a very wide range of total and partial equational specification formalisms can be naturally represented. Whoever makes a proposal of this kind should give good reasons for it. What are the good characteristics of the framework? In which sense is it better than other options? Is it in competition with other proposals?

Of course, statements of optimality can be provisional at best. What is really important is to make clear the intended *goals*, so that the adequacy and success of the proposed formalism to meet such goals can then be judged. In this work the quest is for an equational specification formalism meeting as best as possible the following goals:

1. It should be as *simple* as possible.
2. It should have *all the good properties* of *equational* logics, such as soundness and completeness, initial, free and relatively free algebras, good modularity properties for theory composition and parameterization, and so on.
3. It should be as *general* as possible, so as to encompass the widest possible range of total and partial equational specification formalisms.
4. It should be as *expressive* and as *convenient to use* as possible. In particular, it should directly support features such as
   - sorts and subsorts,
   - operator overloading,
   - errors and error recovery,
   - partiality,

   that are very useful in practical specifications. Therefore, a formalism that would "compile away" such features while still keeping a comparable expressive power at the mathematical level would utterly fail to meet this goal.
5. It should be *efficiently implementable* by rewriting, so that executable specification and declarative programming using this logic become practical and attractive possibilities once the appropriate interpreters and compilers are built. This means that a natural path from specifications to efficient programs exists within the logic. It also means that many other formalisms, once represented in it, can be executed, with enormous savings in tool-building effort.

It should be obvious that the above goals exercise a dynamic and creative *tension* among each other, similar to that of two groups of people pulling a rope in opposite directions, with, say, simplicity, efficient implementability and equational character pulling on one side, and with generality and expressiveness pulling on the other. This makes the task more interesting, because the best

[2] Since the expression "algebraic specification" is now often used in a quite latitudinarian way, so as to mean almost anything that uses an algebraic technique, such as pushouts, in a possibly nonalgebraic setting, I have chosen "equational specification" to mean the hardcore stuff of the equational logics and algebras that we all know and love.

solutions should keep the rope taut, not allowing any particular goal to win at the expense of the others.

Judging whether membership equational logic does indeed meet the above goals is left as an exercise, perhaps the most important exercise, for the reader. It should in any case be pointed out that membership equational logic is in fact very efficiently implementable under usual Church-Rosser assumptions about the specifications. The Maude interpreter [7] can reach up to 590,000 rewrites per second on such specifications running on a Pentium PC, and a compiler would easily achieve several million rewrites per second.

While sharing with CASL its ecumenical leanings, it should be obvious that the goals and criteria involved in each project are quite different, and that they are in fact complementary. Thus, CASL is algebraic in the latitudinarian sense of the word, but, being based on full first-order logic, does not enjoy the usual initiality, freeness and relative freeness properties of equational logics; and it is not reasonable to expect or demand any efficient executability of arbitrary CASL specifications. This is because the goals of generality and expressiveness of CASL's specification formalism completely dominate over those of efficient implementability and equational character, a perfectly legitimate choice for the purposes intended for CASL, but a very different one from those of membership equational logic and Maude. It is however reasonable to expect that a sizable number of CASL specifications will in fact fall within its equational subset, and that one would then desire to benefit from the executability and the powerful automated deduction techniques available in the equational world through appropriate tools. Since the equational logic subset of CASL can be mapped in a conservative way into membership equational logic, this opens up the possibility of using Maude as a tool for CASL, both for execution and—as several of us at SRI are currently doing, by developing a Church-Rosser checker tool and an inductive theorem prover for the Cafe language—as a metatool for building a formal environment of proving tools written in Maude.

Membership equational logic extends in a very natural way both order-sorted algebra with sort constraints [14, 25], and Wadge's classified algebra [34, 33] which can be regarded as the one-kinded[3] subcase of membership equational logic. In addition, membership equational logic is naturally a special case of, and as we shall see in Section 3 actually equivalent to, many-sorted Horn logic with equality, a logic that Joseph Goguen and I studied in detail as the natural setting for unifying equational and relational programming [15]. Therefore, there is nothing particularly surprising about it, except perhaps for how naturally it can be used as an equational logical framework, how well it satisfies the goals listed above, and how long it has taken some of us to recognize it for what it is worth.

The adequacy of membership algebra as a logical framework for equational

[3] In membership equational logic data is first classified in *kinds*, with each kind then having a set of different *sorts*. Kind checking is decidable, whereas sort checking is in general only semidecidable; however, it becomes decidable under adequate Church-Rosser conditions [4, 3].

specification is supported by the fact, discussed in detail in the paper, that—besides the formalisms already mentioned—other well-known total equational approaches to types such as many-sorted equational logic, equational type logic [21], unified algebras [28], and heterogeneous unified algebras [30] can be faithfully embedded in it. Similarly, a wide range of partial equational specification formalisms, including essentially algebraic theories [11, 1], hierarchical equational partial logic [32], left-exact sketches [2], limit theories [10], (conditional) many-sorted partial equational logic with existence equations [32], and partial equational logic with strong equations [5] can all be faithfully embedded in *partial membership equational logic*, a partial analogue of membership equational logic also studied in this paper. Partial membership equational logic can be mapped conservatively—and preserving key model-theoretic constructions such as initial, free, and relatively free algebras—into (total) membership equational logic. In this way, it is shown that membership equational logic has good properties as a logical framework for representing both total and partial equational specification formalisms. In addition, as shown in joint work with Adel Bouhoula and Jean-Pierre Jouannaud [4, 3] membership equational logic enjoys quite attractive automated deduction methods—such as completion and inductive theorem proving—and operational semantics using rewriting techniques. Claus Hintermeier and Claude and Hélène Kirchner have developed similar rewriting and completion techniques in a partial order-sorted setting [17, 18]. However, it seems fair to say that the total approach of membership algebra leads to simpler solutions and methods.

This work should be placed in context by explaining that it is part of a larger collective effort. Membership equational logic was first implemented in the Maude interpreter in joint work with Manuel Clavel, Steven Eker and Patrick Lincoln [7]. Shortly afterwards, Adel Bouhoula, Jean-Pierre Jouannaud and I started a joint effort to investigate in a systematic way its rewriting, proof-theoretic, and automated deduction aspects [4, 3]. At the same time I undertook a systematic study of both its model-theoretic aspects and its properties as an equational logical framework. The fruits of this last study were first reported at Dagstuhl in July of 1996 [23], distributed in long manuscript form during the Asilomar rewriting logic workshop in September of 1996, and, with more advances, presented at WADT'97 in Tarquinia and further spelled out in this version.

The structure of the paper is as follows. After giving the basic definitions in Section 2, we prove in Section 3 that membership equational logic and many-sorted Horn logic with equality define the same classes of models. Then, sound and complete rules of deduction, as well as constructions for initial and free algebras, and for free extensions along theory maps are given in Sections 4–7. A first discussion of membership equational logic as a logical framework for *total* equational specification formalisms is then presented in Section 8. But this raises the question of how, and how well, can partial equational logics be represented. Answering this question properly requires introducing some new technical concepts from the theory of institutions, namely a generalized notion of map of

institutions and the more specific notion of an extension map of institutions in Section 10. The rest of the paper studies in detail two key extension maps of this kind representing order-sorted algebra (Sections 11 and 12) and *partial* membership equational logic (Sections 13 and 14) in membership equational logic. The overall properties of membership equational logic as an equational logical framework in which one can represent a wide range of logics are then discussed in Section 15. The paper finishes with some concluding remarks in Section 16.

2 Basic Definitions

Definition 1. A *signature* Ω in membership equational logic is a triple (K, Σ, π) where K is a set whose elements are called *kinds*, $\Sigma = \{\Sigma_{w,k}\}_{(w,k)\in K^*\times K}$ is a $K^* \times K$-indexed family of function symbols[4], and π is a function $\pi : S \to K$ that assigns to each element of a set S of *sorts* its corresponding kind. We denote by S_k the set $\pi^{-1}(k)$ for $k \in K$.

Note that in the above definition Σ is a K-sorted signature in the usual many-sorted sense. However, since in our terminology the elements of K are called kinds, and they are different from the sorts in S, we will throughout refer to Σ as a *K-kinded* signature, and to its usual many-sorted Σ-algebras as *many-kinded Σ-algebras.*

Definition 2. Given a signature Ω, an *Ω-algebra* consists of

1. a many-kinded Σ-algebra A, together with
2. an assignment to each sort $s \in S$ of a subset $A_s \subseteq A_{\pi(s)}$.

Given two Ω-algebras A and B, an *Ω-homomorphism* $h : A \to B$ is a Σ-homomorphism $h : A \to B$ such that for each sort $s \in S$ we have $h_{\pi(s)}(A_s) \subseteq B_s$.

With the usual composition of homomorphisms this defines a category $\mathbf{Alg}_\Omega$.

In membership equational logic there are two types of *atomic formulas*, namely:

1. *equations* of the form $t = t'$ where $t, t' \in T_\Sigma(X)_k$ for some $k \in K$, for X a K-kinded set of variables and $T_\Sigma(X)$ the free Σ-kinded algebra on X, and
2. *membership assertions* of the form $t : s$, where $s \in S$ and $t \in T_\Sigma(X)_{\pi(s)}$.

[4] As usual, K^* denotes the free monoid on the set K with unit λ. Each $f \in \Sigma_{w,k}$ is denoted $f : w \to k$. Moreover, to avoid ambiguous terms—so that each term has a unique kind—we will assume that Σ satisfies the property that whenever $f \in \Sigma_{w,k} \cap \Sigma_{w',k'}$ with w and w' of same length, then $w = w' \Leftrightarrow k = k'$. Any K-kinded signature has an isomorphic signature with this property. Note that this requirement does not rule out subsort overloading of function symbols. The point is that the same operator at the level of kinds may have several different overloadings at the level of sorts, which, as we shall see, can be expressed as axioms in the logic.

The *sentences* that we shall consider are universally quantified Horn clauses on these atomic formulas, that is, sentences of the form

$$(i)\ \ (\forall X)\ t = t' \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n \wedge w_1 : s_1 \wedge \ldots \wedge w_m : s_m$$
$$(ii)\ \ (\forall X)\ t : s \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n \wedge w_1 : s_1 \wedge \ldots \wedge w_m : s_m$$

where X is a K-kinded set containing all the variables appearing in t, t' (resp. t) and the u_i, v_i, and w_j.

Given an Ω-algebra A and a K-kinded set of variables X such that $t, t' \in T_\Sigma(X)$ (resp. $t \in T_\Sigma(X)$), then *satisfaction* of an atomic formula $t = t'$ (resp. $t : s$) relative to a K-kinded function $a : X \to A$, called an *assignment* of values in A to the variables X, is defined in the obvious way, that is,

$$A, a \models_\Omega t = t' \ \text{ iff } \ \overline{a}(t) = \overline{a}(t')$$
$$A, a \models_\Omega t : s \ \text{ iff } \ \overline{a}(t) \in A_s$$

where $\overline{a} : T_\Sigma(X) \to A$ is the unique K-kinded Σ-homomorphism from $T_\Sigma(X)$ to A as a Σ-algebra extending the assignment a.

Similarly, for φ a Horn sentence of type (i) (resp. (ii)), we say that A *satisfies* φ, written $A \models_\Omega \varphi$, iff for all K-kinded assignments $a : X \to A$ such that $A, a \models_\Omega u_i = v_i$, $1 \leq i \leq n$, and $A, a \models_\Omega w_j : s_j$, $1 \leq j \leq m$, we have $A, a \models_\Omega t = t'$ (resp. $A, a \models_\Omega t : s$).

For Γ a set of Horn sentences we write $A \models_\Omega \Gamma$ iff for all $\varphi \in \Gamma, A \models_\Omega \varphi$.

A *theory* is a pair (Ω, Γ), with Γ a set of Horn Ω-sentences. Such a theory defines a subclass of Ω-algebras, and therefore a full subcategory $\mathbf{Alg}_{\Omega,\Gamma} \subseteq \mathbf{Alg}_\Omega$, namely, exactly those Ω-algebras A such that $A \models_\Omega \Gamma$.

Example 1. The basic features of membership algebra can be illustrated by the following parameterized specification in Maude-like notation[5] for paths in a graph.

```
fth GRAPH is
  sorts Node Edge .
  ops s t : Edge -> Node .
eth

fmod PATH[G :: GRAPH] is
  sorts Path .
  subsorts Node Edge < Path .
  ops s t : Path -> Node .
  op _;_ : [Path] [Path] -> [Path] [assoc] .
  var E : Edge .
  var N : Node .
```

[5] The parameter theory `GRAPH`, introduced by the keyword `fth`, has a *loose* semantics; whereas the parameterized module `PATH`, introduced by the keyword `fmod`, has a *free algebra semantics*, sending each graph to the free `PATH`-algebra that it generates. Such free algebras exist by Theorem 10.

```
  vars P Q : Path .
  cmb E ; P : Path if t(E) == s(P) .
  ceq N ; P = P if s(P) == N .
  ceq P ; N = P if t(P) == N .
  eq s(N) = N .
  eq t(N) = N .
  ceq s(E ; P) = s(E) if t(E) == s(P) .
  ceq t(P ; E) = t(E) if t(P) == s(E) .
efm
```

The above specification is Church-Rosser and terminates with canonical terms that have the least possible sort among all their equivalent terms; therefore, it can be executed in Maude by term rewriting. Note that some syntactic sugar is used in this notation. There is only one kind in the module `PATH`, which is identified with the single connected component {`Node`, `Edge`, `Path`} generated by the subsort ordering, and is denoted by the equivalence class notation `[Path]`. Therefore, strictly speaking all variables are of kind `[Path]`; the sorts declared for the variables are in fact conditions left implicit in the axioms in which the variables appear. For example, the equation `s(N) = N` is syntactic sugar for

```
eq s(N) = N if N : Node .
```

Both subsort declarations and operator declarations are syntactic sugar for the corresponding conditional membership axioms. For example, `Edge < Path` and the operator declaration for the source function `s` mean, respectively,

```
cmb X : Path if X : Edge .
cmb s(X) : Node if X : Path .
```

for `X` a variable of kind `[Path]`. The attribute `[assoc]` of the path composition operator declares that the axiom of associativity holds for it and permits matching and rewriting modulo such an axiom. Path composition is in fact the only operator declared explicitly at the kind level; that the sort and target functions are unary functions of sort `[Path]` can be trivially inferred from their (sorted) operator declarations.

Note that, because of the conditional membership axiom for path composition, in the free `PATH`-algebra generated by a given graph a composition of edges will have sort `Path` if and only if the respective target and source of each pair of contiguous edges coincide. If this condition is not satisfied, such an expression will have kind `[Path]`, but will have no sort assignable to it. That is, it will be an *error expression*, corresponding to the fact that composition of paths is a partial function that is defined exactly when the respective sources and targets coincide.

At parse time we cannot in general determine whether an expression has a sort or not, since this requires performing logical inferences. Therefore, it is possible to give dubious expressions the benefit of the doubt, so that if in the end their canonical form is sortable, we know that the expression is not an error and denotes a well-defined value.

3 The Equivalence with Many-Sorted Horn Logic with Equality

With the above definition of sentences and satisfaction, membership equational logic is indeed an institution[6] [13], that we shall denote *MEqtl*. Furthermore, it is also quite clear that membership equational logic is a special type of Horn formalism and should therefore be viewed as a sublogic [24] of many-sorted Horn logic with equality, denoted $MSHorn^{=}$. That is, there is a sublogic inclusion $I : MEqtl \to MSHorn^{=}$.

Indeed, recall that signatures in $MSHorn^{=}$ are triples (L, Σ, Π) with L a set of sorts, $\Sigma = \{\Sigma_{w,l}\}_{(w,l)\in L^*\times L}$ a family of function symbols, and $\Pi = \{\Pi_w\}_{w\in L^*}$ a family of predicate symbols, and that models are L-sorted Σ-algebras A together with the assignment of a subset $P_A \subseteq A^w$ to each $P \in \Pi_w, w \in L^*$. Then, the inclusion I sends a signature $\Omega = (K, \Sigma, \pi : S \to K)$ to the signature $I(\Omega) = (K, \Sigma, \overline{S})$, where $\overline{S}_k = S_k$ for $k \in K$ and $\overline{S}_w = \emptyset$ for any $w \in K^* - K$, and where for each predicate symbol $s \in S_k$ we adopt the postfix notation $\_ : s$ to match our previous notation.

With this postfix convention the translation of sentences from *MEqtl* to $MSHorn^{=}$ is indeed the identity function, and the corresponding model categories are not only isomorphic, but in fact identical, that is, $\mathbf{Alg}_{\Omega} = \mathbf{Mod}_{I(\Omega)}$ and similarly $\mathbf{Alg}_{\Omega,\Gamma} = \mathbf{Mod}_{I(\Omega),\Gamma}$.

Conversely, we can regard many-sorted Horn logic with equality as a special case of membership equational logic by defining a map of institutions[7] [24] $J : MSHorn^{=} \to MEqtl$. The map J assigns to each signature (L, Σ, Π) the theory whose signature

1. has kind set $K = L \uplus \{p(w) \mid w \in L^* - L \text{ and } \Pi_w \neq \emptyset\}$;
2. for each kind $k \in K$ the set of sorts S_k is
   - if $k \in L$ then $S_k = \Pi_k$,
   - if $k = p(w)$ then $S_{p(w)} = \Pi_w$;
3. the signature of function symbols is

$$\Delta = \Sigma \cup \{\langle \_, \ldots, \_ \rangle : l_1 \ldots l_n \to p(l_1 \ldots l_n) \mid p(l_1 \ldots l_n) \in K - L\}$$
$$\cup \{\pi_i : p(l_1 \ldots l_n) \to l_i, 1 \leq i \leq n \mid p(l_1 \ldots l_n) \in K - L\} \ .$$

   In particular, in the case $\Pi_\lambda \neq \emptyset$, we just have a constant $\langle\rangle : \lambda \to p(\lambda)$ and no projections.

The axioms of the theory $J(L, \Sigma, \Pi)$ are then the equations

$$(\forall x_1 : l_1 \ldots x_n : l_n) \ \ \pi_i(\langle x_1 \ldots x_n \rangle) = x_i, \ \ 1 \leq i \leq n$$
$$(\forall y : p(l_1 \ldots l_n)) \ \ y = \langle \pi_1(y), \ldots, \pi_n(y) \rangle$$

[6] However, the notion of maps of signatures and the corresponding preservation of satisfaction will be dealt with in Section 7.

[7] This map of institutions is furthermore an *embedding* in the sense explained in Section 8.

for each $p(l_1 \ldots l_n) \in K - L$, stating that the kind $p(l_1 \ldots l_n)$ is, up to isomorphism, the cartesian product of the kinds $l_1, \ldots, l_n$ with projection functions $\pi_1, \ldots, \pi_n$. In particular, in the case $p(\lambda) \in K - L$ these equations become the equation $(\forall y : p(\lambda))\ y = \langle\rangle$, stating that the kind $p(\lambda)$ is a one-point set.

The translation α of sentences sends each equation $t = t'$ to itself, each predicate atomic formula $P(t_1, \ldots, t_n)$ to the membership assertion $\langle t_1, \ldots, t_n \rangle : P$, and each Horn clause

$$(\forall X)\ at \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n \wedge P_1(\overline{w_1}) \wedge \ldots \wedge P_m(\overline{w_m})$$

to the Horn sentence

$$(\forall X)\ \alpha(at) \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n \wedge \langle\overline{w_1}\rangle : P_1 \wedge \ldots \wedge \langle\overline{w_m}\rangle : P_m \ .$$

A general Horn theory (L, Σ, Π, Γ) is then mapped by J to the theory obtained by adding to $J(L, \Sigma, \Pi)$ all the axioms $\alpha(\Gamma)$. It is then easy to show that there is an equivalence of categories

$$\mathbf{Mod}_{(L,\Sigma,\Pi)} \approx \mathbf{Alg}_{J(L,\Sigma,\Pi)} \ ,$$

and more generally,

$$\mathbf{Mod}_{(L,\Sigma,\Pi,\Gamma)} \approx \mathbf{Alg}_{J(L,\Sigma,\Pi,\Gamma)} \ .$$

This shows that, as institutions, *MSHorn*$^=$ and *MEqtl* have essentially the same expressive power, since theories can be translated in both directions by maps of institutions, and the corresponding categories of models are equivalent.

4 Rules of Deduction, Soundness and Completeness

The above equivalence with Horn logic also shows that, since we can regard *MEqtl* as a sublogic of many-sorted Horn logic with equality, all the well-known results for many-sorted Horn logic with equality hold true when restricted to the sublogic *MEqtl*. In particular, the proof of soundness and completeness for the rules of deduction of order-sorted Horn logic with equality in [15], of which many-sorted Horn logic with equality is a special case, yields the soundness and completeness of the following rules of deduction for a theory (Ω, Γ) in membership equational logic as an immediate corollary (where all the quantifications are assumed to involve sets of K-kinded variables containing those occurring in all terms of the given sentence).

1. **Reflexivity**.

$$\frac{}{\Gamma \vdash_\Omega (\forall X)\ t = t}$$

2. **Symmetry**.

$$\frac{\Gamma \vdash_\Omega (\forall X)\ t = t'}{\Gamma \vdash_\Omega (\forall X)\ t' = t}$$

3. **Transitivity**.

$$\frac{\Gamma \vdash_\Omega (\forall X)\ t = t' \qquad \Gamma \vdash_\Omega (\forall X)\ t' = t''}{\Gamma \vdash_\Omega (\forall X)\ t = t''}$$

4. **Congruence**.

$$\frac{\Gamma \vdash_\Omega (\forall X)\ t_1 = t_1' \quad \ldots \quad \Gamma \vdash_\Omega (\forall X)\ t_n = t_n'}{\Gamma \vdash_\Omega (\forall X)\ f(t_1, \ldots, t_n) = f(t_1', \ldots, t_n')}$$

   where we assume that $f : k_1 \ldots k_n \to k$ is in Σ, and the terms $t_i, t_i' \in T_\Sigma(X)_{k_i}$, $1 \leq i \leq n$.

5. **Membership**.

$$\frac{\Gamma \vdash_\Omega (\forall X)\ t = t' \qquad \Gamma \vdash_\Omega (\forall X)\ t : s}{\Gamma \vdash_\Omega (\forall X)\ t' : s}$$

6. **Modus ponens**. Given a sentence

$$(\forall X)\ t = t' \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n \wedge w_1 : s_1 \wedge \ldots \wedge w_m : s_m$$
$$\text{(resp. } (\forall X)\ t : s \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n \wedge w_1 : s_1 \wedge \ldots \wedge w_m : s_m\text{)}$$

   in the set Γ of axioms, and given a K-kinded assignment $\theta : X \to T_\Sigma(Y)$, then

$$\frac{\Gamma \vdash_\Omega (\forall Y)\ \overline{\theta}(u_i) = \overline{\theta}(v_i)\ \ 1 \leq i \leq n \qquad \Gamma \vdash_\Omega (\forall Y)\ \overline{\theta}(w_j) : s_j\ \ 1 \leq j \leq m}{\Gamma \vdash_\Omega (\forall Y)\ \overline{\theta}(t) = \overline{\theta}(t') \quad \text{(resp. } (\forall Y)\ \overline{\theta}(t) : s\text{)}}$$

5 Initial and Free Models

Also as an immediate corollary of the existence of initial and free models for order-sorted Horn logic with equality [15], we get both a concrete construction for initial and free algebras as well as their corresponding universal properties.

Theorem 3. *Let X be a K-kinded set, $\Omega = (K, \Sigma, \pi)$ a signature, and Γ a set of Ω-sentences. Then, the quotient $\mathcal{T}_{\Omega,\Gamma}(X)$ of $T_\Sigma(X)$ obtained by imposing the Σ-congruence*[8]

$$t \equiv_{\Gamma(X)} t' \iff \Gamma \vdash_\Omega (\forall X)\ t = t'\ ,$$

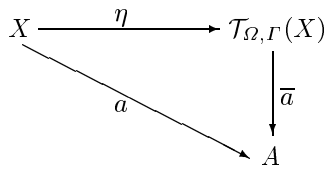
together with the assignment of a subset[9]

$$\mathcal{T}_{\Omega,\Gamma}(X)_s = \{[t] \in \mathcal{T}_{\Omega,\Gamma}(X)_k \mid \Gamma \vdash_\Omega (\forall X)\ t : s\}\ ,$$

is a free (Ω, Γ)*-algebra on X, that is, $\mathcal{T}_{\Omega,\Gamma}(X) \in \mathbf{Alg}_{\Omega,\Gamma}$ and given $A \in \mathbf{Alg}_{\Omega,\Gamma}$, for each assignment $a : X \to A$ there exists a unique Ω-homomorphism $\overline{a} : \mathcal{T}_{\Omega,\Gamma}(X) \to A$ such that the diagram*

[8] This is trivially a Σ-congruence by rules 1–4.

[9] Which does not depend on the choice of a representative by rule 5.



commutes, where η is the assignment mapping each variable x to the equivalence class $[x]$.

Corollary 4. *The algebra $\mathcal{T}_{\Omega,\Gamma} = \mathcal{T}_{\Omega,\Gamma}(\emptyset)$, for $\emptyset$ the empty K-kinded set, is initial in $\mathbf{Alg}_{\Omega,\Gamma}$.*

6 The Theorem of Constants

The rules of deduction 1–6 can only deduce atomic sentences. What about deducing other implications from those in the set of axioms Γ? By viewing variables as extra constants, we can justify the soundness and completeness of adding an implication introduction rule for this purpose to our original rules 1–6.

Given a signature $\Omega = (K, \Sigma, \pi)$ and a K-kinded set X which, without loss of generality, we can assume disjoint from Σ and with $X_k \cap X_{k'} = \emptyset$ if $k \neq k'$, we define $\Omega(X) = (K, \Sigma(X), \pi)$, where $\Sigma(X)$ is the signature obtained by adding the elements of X as constants of the respective kind.

Notice that an Ω-algebra together with an assignment $a : X \to A$, with X a K-kinded set, determines an $\Omega(X)$-algebra structure that can be denoted by the pair (A, a), namely, the extra constants in X are interpreted according to the assignment a. Furthermore, any $\Omega(X)$-algebra can be so described by such a pair (A, a), with A its underlying Ω-algebra structure. Notice also that, by the definition of satisfaction, $A, a \models_\Omega \ t = t'$ (resp. $t : s$) iff $(A, a) \models_{\Omega(X)} (\forall \emptyset)\ t = t'$ (resp. $(\forall \emptyset)\ t : s$).

Let now

$$\varphi = (\forall X)\ t = t' \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n \wedge w_1 : s_1 \wedge \ldots \wedge w_m : s_m$$
$$(\text{resp. } \varphi = (\forall X)\ t : s \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n \wedge w_1 : s_1 \wedge \ldots \wedge w_m : s_m)$$

be a Horn sentence. Can we characterize when all algebras for a theory (Ω, Γ) satisfy such a sentence?

By reinterpreting the definition of satisfaction as satisfaction for the corresponding $\Omega(X)$-algebras, we can do this as follows:

For all $A \in \mathbf{Alg}_{\Omega,\Gamma}$, $A \models_\Omega \varphi$

iff

For all $(A, a) \in \mathbf{Alg}_{\Omega(X),\Gamma}$, if $(A, a) \models_{\Omega(X)} (\forall \emptyset)\ u_i = v_i$, $1 \leq i \leq n$ and $(A, a) \models_{\Omega(X)} (\forall \emptyset)\ w_j : s_j$, $1 \leq j \leq m$, then $(A, a) \models_{\Omega(X)} (\forall \emptyset)\ t = t'$ (resp. $(A, a) \models_{\Omega(X)} (\forall \emptyset)\ t : s$)

iff

For all $(A, a) \in \mathbf{Alg}_{\Omega(X),\Gamma \cup \{(\forall \emptyset)\ u_i = v_i\}_i \cup \{(\forall \emptyset)\ w_j : s_j\}_j}$ we have $(A, a) \models_{\Omega(X)} (\forall \emptyset)\ t = t'$ (resp. $(A, a) \models_{\Omega(X)} (\forall \emptyset)\ t : s$)

iff (by completeness)
$\Gamma \cup \{(\forall \emptyset) u_i = v_i\}_i \cup \{(\forall \emptyset) w_j : s_j\}_j \vdash_{\Omega(X)} (\forall \emptyset)\ t = t'$ (resp. $(\forall \emptyset)\ t : s$) .
As an immediate corollary of this discussion we have

Theorem 5. *The rules of deduction 1–6 together with the rule*

7. **Implication introduction**.

$$\frac{\Gamma \cup \{(\forall \emptyset)\ u_i = v_i\}_i \cup \{(\forall \emptyset\ w_j : s_j\}_j \vdash_{\Omega(X)} (\forall \emptyset)\ t = t'}{\Gamma \vdash_\Omega (\forall X)\ t = t' \Leftarrow \bigwedge_i u_i = v_i \wedge \bigwedge_j w_j : s_j}$$

$$\frac{\Gamma \cup \{(\forall \emptyset)\ u_i = v_i\}_i \cup \{(\forall \emptyset)\ w_j : s_j\}_j \vdash_{\Omega(X)} (\forall \emptyset)\ t : s}{\Gamma \vdash_\Omega (\forall X)\ t : s \Leftarrow \bigwedge_i u_i = v_i \wedge \bigwedge_j w_j : s_j}$$

are sound and complete for deducing all Horn sentences that are logical consequences of some given set of Horn sentences Γ, where now we allow some of the deductions to take place in theories extending (Ω, Γ) by additional constants and additional hypotheses.

7 Free Extensions along Theory Maps

We now define signature morphisms and theory morphisms, and show that membership equational logic is a liberal institution.

Definition 6. Given two signatures $\Omega = (K, \Sigma, \pi)$ and $\Omega' = (K', \Sigma', \pi')$, a *signature morphism* $H : \Omega \to \Omega'$ is a triple $H = (H_0, H_1, H_2)$, where $H_0 : K \to K'$ and $H_1 : S \to S'$ are functions such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{H_1} & S' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ K & \xrightarrow[H_0]{} & K' \end{array}$$

commutes, and where H_2 is a $K^* \times K$-indexed function

$$H_2 = \{H_{2,w,k} : \Sigma_{w,k} \to \Sigma'_{H_0(w), H_0(k)}\}_{(w,k) \in K^* \times K}$$

with $H_0(w)$ the homomorphic extension of H_0 to strings in K^*.

We can therefore describe signature morphisms $(H_0, H_1, H_2) : (K, \Sigma, \pi) \longrightarrow (K', \Sigma', \pi')$ as ordinary many-kinded signature morphisms $(H_0, H_2) : (K, \Sigma) \longrightarrow (K', \Sigma')$ together with a map $H_1 : S \longrightarrow S'$ commuting the square above.

Note that a theory map $H : \Omega \to \Omega'$ induces a forgetful functor

$$\mathcal{U}_H : \mathbf{Alg}_{\Omega'} \to \mathbf{Alg}_\Omega$$

sending each algebra $A' \in \mathbf{Alg}_{\Omega'}$ to the Ω-algebra $\mathcal{U}_H(A')$ with $\mathcal{U}_H(A')_k = A'_{H_0(k)}$, $\mathcal{U}_H(A')_s = A'_{H_1(s)}$, and for each $f : w \to k$ in Σ, $\mathcal{U}_H(A')_f = A'_{H_2(f)}$.

In particular, for X a K-kinded set with $X_{k_1} \cap X_{k_2} = \emptyset$ if $k_1 \neq k_2$, denoting by $H_0(X)$ the K'-kinded set with $H_0(X)_{k'} = \bigcup_{H_0(k)=k'} X_k$, we can consider the algebra $\mathcal{U}_H(\mathcal{T}_{\Omega'}(H_0(X)))$ and define the K-kinded assignment

$$j : X \to \mathcal{U}_H(\mathcal{T}_{\Omega'}(H_0(X)))$$

sending each variable $x \in X_k$ to itself in $H_0(X)_{H_0(k)} \subseteq \mathcal{T}_{\Omega'}(H_0(X))_{H_0(k)} = \mathcal{U}_H(\mathcal{T}_{\Omega'}(H_0(X)))_k$. By the freeness of $\mathcal{T}_\Omega(X)$, j then extends uniquely to an Ω-homomorphism

$$H(X) : \mathcal{T}_\Omega(X) \to \mathcal{U}_H(\mathcal{T}_{\Omega'}(H_0(X)))$$

which provides the obvious translation from Ω-terms to Ω'-terms induced by H. We can extend this translation to Horn sentences by defining

$$\begin{aligned} &H((\forall X)\ t = t' \ \ (\text{resp. } t : s) \Leftarrow \textstyle\bigwedge_i u_i = v_i \wedge \bigwedge_j w_j : s_j) = \\ &(\forall H_0(X))\ H(X)(t) = H(X)(t') \ \ (\text{resp. } H(X)(t) : H_1(s)) \Leftarrow \\ &\textstyle\bigwedge_i H(X)(u_i) = H(X)(v_i) \wedge \bigwedge_j H(X)(w_j) : H_1(s_j) \ . \end{aligned}$$

We leave for the reader to check the following

Lemma 7 (Satisfaction Lemma). *Let $H : \Omega \to \Omega'$ be a signature morphism and let φ be a Horn Ω-sentence. Then, for each algebra $A' \in \mathbf{Alg}_{\Omega'}$ we have*

$$\mathcal{U}_H(A') \models_\Omega \varphi \iff A' \models_\Omega H(\varphi) \ .$$

Theory morphisms are then defined in the usual way.

Definition 8. Given two theories (Ω, Γ) and (Ω', Γ'), a *theory morphism* $H : (\Omega, \Gamma) \to (\Omega', \Gamma')$ is a signature morphism $H : \Omega \to \Omega'$ such that for each $\varphi \in \Gamma$ we have $\Gamma' \vdash_{\Omega'} H(\varphi)$.

Signature morphisms compose in the obvious way and form a category **Sign**. Similarly, theory morphisms compose as well, forming a category **Th**. Furthermore, by a general result about institutions, the category **Th** is finitely cocomplete iff **Sign** is so. Indeed, we have

Lemma 9. **Sign** *and therefore* **Th** *are finitely cocomplete.*

Proof sketch. The initial object in **Sign** is the signature $\emptyset = (\emptyset, \emptyset, 1_\emptyset)$. To see that **Sign** has pushouts, notice that many-kinded signatures have pushouts, and obtain the function π for the pushout as the unique function from the pushout of the sorts to the pushout of the kinds. □

Thanks to the satisfaction lemma, a theory morphism $H : (\Omega, \Gamma) \to (\Omega', \Gamma')$ induces a forgetful functor

$$\mathcal{U}_H : \mathbf{Alg}_{\Omega', \Gamma'} \to \mathbf{Alg}_{\Omega, \Gamma} \ .$$

We now show that membership equational logic is a *liberal* institution in the sense that any such $\mathcal{U}_H$ always has a left adjoint.

Theorem 10. *For any theory morphism $H : (\Omega, \Gamma) \to (\Omega', \Gamma')$ the functor $\mathcal{U}_H$ has a left adjoint $\mathcal{F}_H$.*

Proof sketch. Given $A \in \mathbf{Alg}_{\Omega,\Gamma}$, let $Diag_H(A)$ be the following set of ground $\Omega'(H_0(A))$-sentences:

1. $(\forall \emptyset)\ H(f)(a_1, \ldots, a_n) = A_f(a_1, \ldots, a_n)$
   for each $f : k_1 \ldots k_n \to k$ in Σ, and $a_i \in A_{k_i}$, $1 \leq i \leq n$, where $A_f(a_1, \ldots, a_n)$ denotes the result of the operation f applied to $a_1, \ldots, a_n$ in the algebra A.
2. $(\forall \emptyset)\ a : H_1(s)$ for each $s \in S$ and $a \in A_s$ in A.

Then define $\mathcal{F}_H(A)$ as the underlying Ω'-algebra of the initial algebra

$$\mathcal{T}_{\Omega'(H_0(A)),\Gamma' \cup Diag_H(A)} \ .$$

□

8 Comparison with Some Total Equational Logics

The claim that membership equational logic has good properties as a logical framework for equational specification has to be substantiated in practice by showing how other equational specification formalisms can be represented inside membership equational logic. To begin with, we should consider *total* equational specification formalisms; partial specification formalisms will be discussed later, in Section 15.

The total equational specification formalisms that we shall consider constitute what might be called a representative collection of such formalisms, in the sense that, although not being an exhaustive list of the formalisms that have been proposed, seems to cover the most commonly used ones[10]. In all cases, the representation maps translating each formalism into membership equational logic are *embeddings* of institutions, that is, a particular kind of maps of institutions generalizing sublogic inclusions [24] and having especially nice properties. We assume familiarity with the notion of maps of institutions in [24] and give here a definition of embedding maps that generalizes one due to Mossakowski [27].

Definition 11. A map of institutions $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ is called an *embedding* iff for each $T \in \mathbf{Th}_{\mathcal{I}}$ the functor $\beta_T : Mod'(\Phi(T)) \to Mod(T)$ is an equivalence of categories. We then use the notation $(\Phi, \alpha, \beta) : \mathcal{I} \hookrightarrow \mathcal{I}'$ to denote an embedding map. It is easy to show that embedding maps are closed under composition.

Remark that Φ maps theories in the first institution into theories in the second. As it is well-known for maps of institutions, this allows more flexible mappings than those mapping signatures to signatures.

[10] For a good survey of equational specification formalisms see [29].

Mossakowski [27] has generalized his notion of embedding to that of *weak embedding*, denoted $\mathcal{I} \hookleftarrow\!\!\!\!- \to \mathcal{I}'$. We shall encounter such weak embeddings later, in Section 15.

Embedding maps $\mathcal{I} \hookrightarrow \mathcal{I}'$ have very good properties. If $\mathcal{I}$ and $\mathcal{I}'$ are liberal institutions, they preserve relatively free model constructions, and if isomorphic models in $\mathcal{I}$ satisfy the same sentences then, assuming that $\mathcal{I}$ and $\mathcal{I}'$ have entailment systems that are complete relative to their satisfaction relations, the embedding is a *conservative* map of logics. All this follows from more general results in Section 10. In summary, an embedding $\mathcal{I} \hookrightarrow \mathcal{I}'$ indicates that $\mathcal{I}$ can be regarded as essentially a sublogic of $\mathcal{I}'$, but gives more syntactic flexibility than a sublogic inclusion by allowing maps Φ from theories to theories, and by not requiring the translation of sentences to be an inclusion, or even injective.

Note that we have already encountered in Section 3 embeddings $\mathit{MSHorn}^{=} \hookrightarrow \mathit{MEqtl}$ and $\mathit{MEqtl} \hookrightarrow \mathit{MSHorn}^{=}$, demonstrating that both logics have essentially the same expressive power.

Since *unsorted* Horn logic with equality $\mathit{USHorn}^{=}$ is a special case sublogic of many-sorted Horn logic with equality, we obtain by composition an embedding

$$\mathit{USHorn}^{=} \hookrightarrow \mathit{MSHorn}^{=} \hookrightarrow \mathit{MEqtl} \ .$$

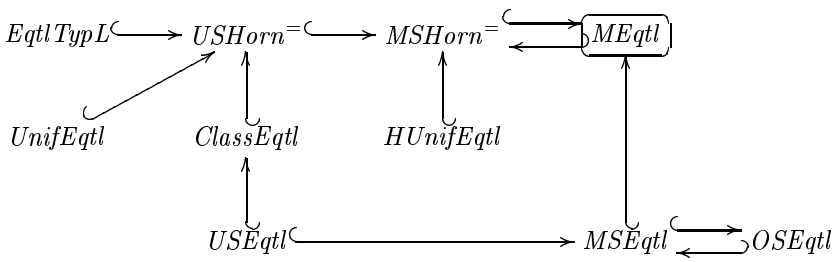
A number of important total equational specification formalisms are naturally embedded in $\mathit{USHorn}^{=}$, often as sublogics, including the following:

- Unsorted (conditional) equational logic *USEqtl*.
- The *equational type logic* (*EqtlTypL*) of Manca, Sallibra and Scollo [21], which is the special case of unsorted Horn logic with equality obtained by allowing only a binary "typing" predicate $\_ : \_$ besides equality.
- As already pointed out in the introduction, the *classified algebra* (*ClassEqtl*) of Wadge [34, 33] is the sublogic of membership equational logic obtained by allowing only a single kind. In spite of its very good properties, such a restriction to a single kind makes classified algebra strictly more limited in expressive power—it cannot for example represent the usual many-sorted formalisms by means of embeddings. Furthermore, the useful type checking allowed by many-kinded specifications, by which we can throw away nonsensical expressions such as `true/7` that do not have a parse at the kind level, is not available in the one-kinded case.
- The *unified algebra* formalism (*UnifEqtl*) of Mosses [28] can also be regarded as essentially a sublogic of unsorted Horn logic with equality. Indeed, Mosses [28] defines in detail an embedding $\mathit{UnifEqtl} \hookrightarrow \mathit{USHorn}^{=}$ showing that the models of a unified algebra specification are precisely the models of a corresponding theory in unsorted Horn logic with equality having a typing predicate $\_ : \_$ and a partial order predicate $\_ \leq \_$ in addition to the equality predicate, and satisfying suitable axioms that characterize the lattice-theoretic and typing properties of unified algebras. Similarly, the *heterogeneous* version of unified algebras (*HUnifEqtl*) proposed by Parisi-Presicce and Veglioni [30] has an embedding into into many-sorted Horn logic with

equality $HUnifEqtl \hookrightarrow MSHorn^{=}$, provided that the signature morphisms are restricted to be injective (see [30, Props. 18–21]).

Of course, the well-known case of (conditional) many-sorted equational logic (*MSEqtl*) is also a sublogic of membership equational logic, namely the sublogic obtained by making the set S of sorts empty. Finally, it is also well-known that (conditional) many-sorted equational logic is a sublogic of (conditional) order-sorted equational logic (*OSEqtl*) as formulated in [16], namely the case when the poset of sorts is discrete, and that, conversely, there is also an embedding of logics $OSEqtl \hookrightarrow MSEqtl$ [16], which composed with the embedding $MSEqtl \hookrightarrow MEqtl$ gives us an embedding $OSEqtl \hookrightarrow MEqtl$. However, as we shall see later (in Sections 11 and 12), this is not the only possible—nor the most natural—way of representing order-sorted equational logic in membership equational logic.

We can summarize our present discussion by means of the following diagram of embeddings:



9 A Partiality Puzzle

The very fact that membership equational logic has the same expressive power as many-sorted Horn logic with equality seems to point to a fundamental limitation, and to cast serious doubts about its adequacy as a logical framework for equational specification. After all, from a model-theoretic and categorical characterization of categories of models defined by increasingly more expressive logics we know that:

1. Each *variety*, that is, each category of many-sorted algebras definable by a set of unconditional equations, is a special case of a *semivariety*, that is, a category of many-sorted algebras definable by conditional equations, but that there are semivarieties that cannot be specified by unconditional equations alone in the very strong sense that they can never be equivalent as categories to a variety, regardless of how such a variety is specified. A simple such counterexample is the category of graphs (as sets endowed with a binary relation) and graph homomorphisms [1, 3.21.2].
2. Similarly, each semivariety is a category of many-sorted models definable in Horn logic with equality but the converse is not true. A well-known counterexample is the category of partially ordered sets [1, 3.21.1].

3. The real trouble comes with *partial semivarieties*, that is, categories of partial algebras definable by conditional equations. Indeed, such categories are exactly the finitely locally presentable categories (see [1, Theorem 3.36] and Section 13) but there are finitely locally presentable categories that are not Horn definable [1, Example 5.13].

In summary, we have

$$\textit{Varieties} \subsetneq \textit{Semivarieties} \subsetneq \textit{Horn}^{=}\textit{-Classes} \subsetneq \textit{Partial Semivarieties}$$

and it is clear that membership equational logic can specify, up to equivalence of categories, exactly the categories of many-sorted models definable by Horn clauses with equality. Therefore, doesn't it then follow that membership equational logic, while clearly adequate as a logical framework for total algebras, is also clearly inadequate as a logical framework for partial ones?

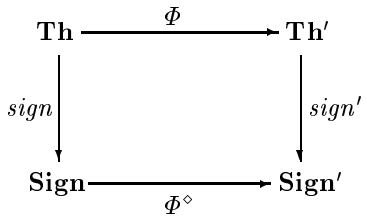
The answer to this puzzle is a rotund *no*. It only shows that (conditional) partial equational logics cannot be mapped into *MEqtl* by means of *embeddings*. But what seems at first sight a limitation is actually the strongest and most attractive feature of membership equational logic as a logical framework, namely:

1. Its logic and model theory are quite simple, in fact strictly simpler than the more complex logic and model categories of (conditional) partial algebras, yet
2. This more complex world of partiality can be faithfully embedded in the simpler world of membership equational logic by means of *extension maps of institutions* such that:
   (a) they leave intact the categories of partial algebras, embedding them in categories of membership algebras, from which they can be recovered back by an adjoint construction,
   (b) they preserve all free constructions in both directions, so that for all practical purposes we can reason in the simpler world of membership algebras,
   (c) the simpler logic of membership algebra can be *borrowed* to endow the corresponding categories of partial algebras with *sound* and *complete* rules of logical deduction in the framework logic.

10 Extension Maps of Institutions

Therefore, before discussing further the logical framework properties of *MEqtl* we introduce some new general notions in the theory of institutions, namely, generalized maps of institutions and extension maps; and we show that extension maps enjoy very good properties.

Definition 12. Given two institutions $\mathcal{I}$ and $\mathcal{I}'$, a functor $\Phi : \mathbf{Th} \to \mathbf{Th}'$ is called *signature-preserving* iff there is a functor $\Phi^{\diamond} : \mathbf{Sign} \to \mathbf{Sign}'$ such that the diagram



commutes. It is then easy to see that $\Phi^\diamond$ is necessarily unique.

The following definition of generalized maps of institutions provides a more general and flexible way of mapping one institution to another than the usual maps of institutions in [24]. This extra generality will in fact be needed in Section 14 to map partial algebras into membership algebras. The extra generality consists in allowing one sentence in the first institution to be translated into a *set* of sentences in the second, and in relaxing the natural transformation requirement for the translation of sentences α to a weaker condition on the corresponding categories of models.

Definition 13. Given two institutions $\mathcal{I}$ and $\mathcal{I}'$, a *generalized map of institutions* $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ is given by

- a signature-preserving functor $\Phi : \mathbf{Th} \to \mathbf{Th}'$,
- a family of functions $\alpha = \{\alpha_\Sigma : sen(\Sigma) \to \mathcal{P}(sen'(\Phi^\diamond(\Sigma)))\}_{\Sigma \in \mathbf{Sign}}$, and
- a natural transformation $\beta : Mod \circ \Phi^{op} \Rightarrow Mod$,

such that:

(i) Denoting $\Phi(\Sigma, \Gamma) = (\Sigma', \Gamma')$ and $\Phi(\Sigma, \emptyset) = (\Sigma', \emptyset'_\Sigma)$, we have

$$Mod'(\Sigma', \Gamma') = Mod'(\Sigma', \emptyset'_\Sigma \cup \bigcup \alpha_\Sigma(\Gamma)) \ .$$

(ii) For each $H : \Sigma_1 \to \Sigma_2$ in **Sign**, and each $\varphi \in sen(\Sigma_1)$, we have

$$Mod'(\Sigma'_2, \emptyset'_{\Sigma_2} \cup \alpha_{\Sigma_2}(H(\varphi))) = Mod'(\Sigma'_2, \emptyset'_{\Sigma_2} \cup \Phi^\diamond(H)(\alpha_{\Sigma_1}(\varphi))) \ .$$

(iii) For each $\Sigma \in \mathbf{Sign}$, $\varphi \in sen(\Sigma)$, and $M' \in Mod'(\Sigma', \emptyset'_\Sigma)$, we have

$$M' \models'_{\Sigma'} \alpha_\Sigma(\varphi) \iff \beta_{(\Sigma,\emptyset)}(M') \models_\Sigma \varphi \ .$$

Where we have shortened $sen(H)$ to H, and $sen'(\Phi^\diamond(H))$ to $\Phi^\diamond(H)$.

It is not hard to show that generalized maps of institutions form a category that contains the usual maps of institutions as a subcategory.

The following proposition gives a particularly compact way of defining a generalized map of institutions.

Proposition 14. *Given two institutions $\mathcal{I}$ and $\mathcal{I}'$, a functor $\Phi_0 : \mathbf{Sign} \to \mathbf{Th}'$ (where we then denote by $\Phi^\diamond : \mathbf{Sign} \to \mathbf{Sign}'$ the functor $\Phi^\diamond = sign' \circ \Phi_0$), a family of functions $\alpha = \{\alpha_\Sigma : sen(\Sigma) \to \mathcal{P}(sen'(\Phi^\diamond(\Sigma)))\}_{\Sigma \in \mathbf{Sign}}$, and a natural transformation $\beta : Mod \circ \Phi_0^{op} \Rightarrow Mod$, satisfying conditions (ii) (with the slight change of $\beta_{(\Sigma,\emptyset)}$ by β_Σ) and (iii), can be extended[11] in a unique way to a generalized map of institutions $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ whose restriction to signatures is $\Phi^\diamond$ and such that if $\Phi_0(\Sigma) = (\Sigma', \emptyset'_{\Sigma'})$, then $\Phi(\Sigma, \Gamma) = (\Sigma', \emptyset'_{\Sigma'} \cup \bigcup \alpha_\Sigma(\Gamma))$.*

We are now ready to define extension maps.

Definition 15. A generalized map of institutions $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ is called an *extension* iff for each $T \in \mathbf{Th}_\mathcal{I}$ the functor $\beta_T : Mod'(\Phi(T)) \to Mod(T)$ has a left adjoint ρ_T such that the unit of the adjunction is a natural isomorphism

$$1_{\mathrm{Mod}} \simeq \beta_T \circ \rho_T \quad .$$

By general results of category theory [20], the functor ρ is then full and faithful. The functor ρ_T is called the *extension* functor, whereas β_T is called the *restriction* functor. Intuitively, the models of $Mod'(\Phi(T))$ contain those of $Mod(T)$ as a subclass, and can be "restricted" to that subclass via β.

We will use the notation $(\Phi, \alpha, \beta) : \mathcal{I} \rightarrowtail \mathcal{I}'$ to denote an extension map of institutions. It is easy to show that extension maps are closed under composition, and that they contain embeddings as a subcategory.

I am indebted to Till Mossakowski for pointing out that the above notion of extension map is closely related to his notion of *categorical retractive simulation*, which is a map of institutions (Φ, α, β) with a right-inverse left adjoint for each β_T [19]. In fact, categorical retractive simulations are a special case of the more general notion of an extension map.

The following proposition shows that if an institution $\mathcal{I}'$ has an entailment system making it into a complete logic, then we can use an extension map $\mathcal{I} \rightarrowtail \mathcal{I}'$ to "borrow" such an entailment system by the general method of [6] to make $\mathcal{I}$ a complete logic. We will later use this result with $\mathcal{I}' = MEqtl$ to endow other logics with sound and complete rules of deduction.

Proposition 16. *Let $(\Phi, \alpha, \beta) : \mathcal{I} \rightarrowtail \mathcal{I}'$ be an extension map such that isomorphic models in $\mathcal{I}$ satisfy the same sentences, and such that $\vdash'$ is a complete entailment system for $\mathcal{I}'$. Then, defining*

$$\Gamma \vdash_\Sigma \varphi \quad \Longleftrightarrow \quad \text{for all } \psi \in \alpha_\Sigma(\varphi),\ \ \emptyset'_{\Sigma'} \cup \bigcup \alpha_\Sigma(\Gamma) \vdash'_{\Sigma'} \psi$$

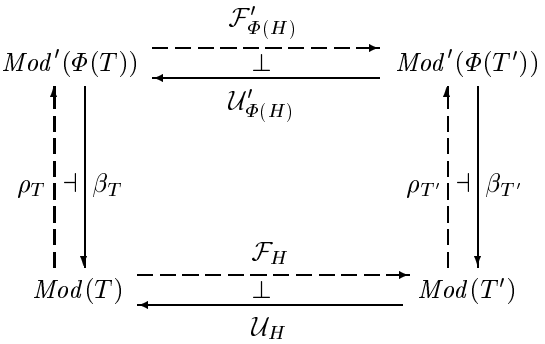
yields a complete entailment system for $\mathcal{I}$.

Proof. We need to show that $\Gamma \vdash_\Sigma \varphi$ iff for all $M \in Mod(\Sigma, \Gamma), M \models_\Sigma \varphi$. By (Φ, α, β) being an extension map and isomorphic models in $\mathcal{I}$ satisfying the same sentences we have

[11] In the obvious sense of extension.

for all $M \in Mod(\Sigma, \Gamma), M \models_\Sigma \varphi$
iff
for all $M' \in Mod'(\Phi(\Sigma, \Gamma)),\ \beta(M') \models_\Sigma \varphi$
iff (by condition (iii))
for all $M' \in Mod'(\Phi(\Sigma, \Gamma))$, for all $\psi \in \alpha_\Sigma(\varphi),\ M' \models_{\Sigma'} \psi$
iff (by $\vdash'$ a complete entailment system and condition (i))
for all $\psi \in \alpha_\Sigma(\varphi),\ \emptyset'_\Sigma \cup \bigcup \alpha_\Sigma(\Gamma) \vdash'_{\Sigma'} \psi$. □

Theorem 17. *Let $(\Phi, \alpha, \beta) : \mathcal{I} \rightarrowtail \mathcal{I}'$ be an extension map of institutions with $\mathcal{I}$ and $\mathcal{I}'$ liberal. Then, free constructions are preserved by both extension and restriction in the sense that in the following diagram of adjoints associated to a theory morphism $H : T \to T'$*



we have for each $M \in Mod(T)$

$$(i) \quad \mathcal{F}'_{\Phi(H)}(\rho_T(M)) \simeq \rho_{T'}(\mathcal{F}_H(M))$$
$$(ii) \quad \mathcal{F}_H(M) \simeq \beta_{T'}(\mathcal{F}'_{\Phi(H)}(\rho_T(M))) \ .$$

Proof. (i) follows immediately from the commutativity of the square of solid arrows by naturality of β, plus composition of adjoints being an adjoint, and adjoints being unique up to isomorphism. To see (ii), notice that

$$\beta_{T'}(\mathcal{F}'_{\Phi(H)}(\rho_T(M))) \simeq \beta_{T'}(\rho_{T'}(\mathcal{F}_H(M))) \simeq \mathcal{F}_H(M)$$

using first (i) and then that (Φ, α, β) is an extension. □

We will apply the notion of extension map and its properties to define maps into *MEqtl* in two important cases, namely order-sorted algebra (in Sections 11 and 12), and partial membership algebra (in Sections 13 and 14).

11 Order-Sorted Algebra Revisited

We should map to *MEqtl* the "best possible version" of order-sorted algebra (OSA). There are good reasons to believe that the versions now used like OSA^{GM} of Goguen and Meseguer [16] and OSA^{P} of Poigné [31] are unsatisfactory. I

propose here a new version, $\mathrm{OSA}^{\mathrm{R}}$, that combines all the advantages and does not seem to have any drawbacks.

The following table summarizes some key desirable properties and how they are met, or fail to be met, by each variant of order-sorted algebra. The yes/no answers would have to be qualified in some cases, but this will have to wait for a longer discussion. Note that $\mathrm{OSA}^{\mathrm{GM}}$ and $\mathrm{OSA}^{\mathrm{P}}$ are Boolean complements of each other with respect to these properties.

	$\mathrm{OSA}^{\mathrm{GM}}$	$\mathrm{OSA}^{\mathrm{P}}$	$\mathrm{OSA}^{\mathrm{R}}$
Contains MSA as special case	Yes	No	Yes
Term algebras initial with minimal requirements	No	Yes	Yes
Pushouts of theories exist in general	No	Yes	Yes
Ad-hoc overloading possible	Yes	No	Yes
Terms of different sorts allowed in equations	Yes	No	Yes
Natural conservative extension map to *MEqtl*	No	Yes	Yes

Definition 18. An *order-sorted signature* is a triple $\Sigma = (S, \leq, \Sigma)$ with $S = (S, \leq)$ a poset and (S, Σ) a many-sorted signature.

We denote by $\hat{S} = S/\equiv_{\leq}$ the set of *connected components* of $(S, \leq)$, which is the quotient of S under the equivalence relation $\equiv_{\leq}$ generated by $\leq$. The equivalence $\equiv_{\leq}$ can be extended to sequences in the usual way.

The notion of a sensible signature is a minimal syntactic requirement to avoid excessive ambiguity. However, it permits fully flexible ad-hoc and subsort overloading of function symbols.

Definition 19. An order-sorted signature is called *sensible* if for any two operators $f : w \to s, f : w' \to s'$, with w and w' of same length, we have $w \equiv_{\leq} w' \Leftrightarrow s \equiv_{\leq} s'$.

Any order-sorted signature can be transformed into an isomorphic signature that is sensible and still has the desired ad-hoc and subsort overloadings.[12]

Notation. For connected components $k_1, \ldots, k_n, k \in \hat{S}$

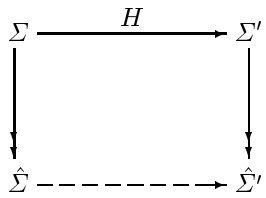
$$f_k^{k_1 \ldots k_n} = \{f : s_1 \ldots s_n \to s \mid s_i \in k_i \;\; 1 \leq i \leq n, \; s \in k\}$$

denotes the family of "subsort polymorphic" operators with name f for those components.

Using the projection to the quotient $S \twoheadrightarrow \hat{S}$ we can define a many-sorted signature $(\hat{\Sigma}, \hat{S})$ with $f : k_1 \ldots k_n \to k$ iff $f_k^{k_1 \ldots k_n} \neq \emptyset$ in Σ.

[12] One can decorate each $f : s_1 \ldots s_n \longrightarrow s$ with the string of connected components $[s_1] \ldots [s_n][s]$ to get operators $f_{[s_1]\ldots[s_n][s]} : s_1 \ldots s_n \longrightarrow s$. Ad-hoc overloading can then be recovered by making suitable identifications between decorated function symbols without violating sensibility.

Definition 20. *Signature morphisms* $H : (S, \leq, \Sigma) \to (S', \leq', \Sigma')$ are many-sorted signature morphisms $H : (S, \Sigma) \to (S', \Sigma')$ with $H : (S, \leq) \to (S', \leq')$ monotonic and such that they preserve subsort overloading, that is, such that the following diagram commutes:



Theorem 21. *The category* **OSSign** *of order-sorted signatures and homomorphisms is cocomplete.*

Definition 22. For $\Sigma = (S, \leq, \Sigma)$ an order-sorted signature, an *order-sorted* Σ*-algebra* A is a many-sorted (S, Σ)-algebra A such that

- whenever $s \leq s'$, then we have $A_s \subseteq A_{s'}$, and
- whenever $f : w \to s, f : w' \to s'$ in $f_k^{k_1 \ldots k_n}$ and $\overline{a} \in A^w \cap A^{w'}$, then we have $A_{f:w \to s}(\overline{a}) = A_{f:w' \to s'}(\overline{a})$.

An *order-sorted* Σ*-homomorphism* $h : A \to B$ is a many-sorted (S, Σ)-homomorphism such that whenever $s, s' \in k$ and $a \in A_s \cap A_{s'}$, then we have $h_s(a) = h_{s'}(a)$.

This defines a category $\mathbf{OSAlg}_\Sigma$.

Notice that, given an order-sorted Σ-algebra A, if we denote by $A_k = \bigcup_{s \in k} A_s$ for each $k \in \hat{S}$, then each $f_k^{k_1 \ldots k_n}$ determines a single *partial* function

$$A_{f_k^{k_1 \ldots k_n}} : A_{k_1} \times \cdots \times A_{k_n} \multimap A_k \ .$$

Similarly, an order-sorted Σ-homomorphism $h : A \to B$ determines an $\hat{S}$-family of functions

$$\{h_k : A_k \to B_k\}_{k \in \hat{S}} \ .$$

Theorem 23. *The category* $\mathbf{OSAlg}_\Sigma$ *has an initial algebra. Furthermore, if* Σ *is sensible, then the term algebra* $\mathcal{T}_\Sigma$ *with*

- *if* $a : \lambda \to s$ *then* $a \in \mathcal{T}_{\Sigma,s}$,
- *if* $t \in \mathcal{T}_{\Sigma,s}$ *and* $s \leq s'$ *then* $t \in \mathcal{T}_{\Sigma,s'}$,
- *if* $f : s_1 \ldots s_n \to s$ *and* $t_i \in \mathcal{T}_{\Sigma,s_i}$ $1 \leq i \leq n$, *then* $f(t_1, \ldots, t_n) \in \mathcal{T}_{\Sigma,s}$,

is initial.

Variable declarations have the form

$$\tilde{X} = \{x_1 : s_1^1, \ldots, s_{l_1}^1; \ldots; x_m : s_1^m, \ldots, s_{l_m}^m\}$$

with $s^i_1, \ldots, s^i_{l_i} \in k_i \;\; 1 \leq i \leq m$, and then we can consider terms in the free algebra $\mathcal{T}_\Sigma(\tilde{X})$, which can be constructed as the initial algebra for the signature $\Sigma(\tilde{X})$ in which the variables x_i are added disjointly to Σ as constants with sorts s^i_j.

Conditional order-sorted Σ-equations are of the form

$$(\forall \tilde{X})\ t = t' \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n$$

with $t, t' \in \mathcal{T}_\Sigma(\tilde{X})_k$, and $u_i, v_i \in \mathcal{T}_\Sigma(\tilde{X})_{k_i} \;\; 1 \leq i \leq n$.

Satisfaction of such a conditional equation by an order-sorted Σ-algebra A can then be defined in the usual way using sort-preserving assignments $a : \tilde{X} \to A$.

An *order-sorted theory* is then a pair (Σ, Γ) with Γ a set of conditional Σ-equations. The category $\mathbf{OSAlg}_{\Sigma,\Gamma}$ of algebras for such a theory is the full subcategory of $\mathbf{OSAlg}_\Sigma$ defined by those algebras that satisfy the axioms in Γ.

Given a sensible order-sorted signature $(S, \leq, \Sigma)$ and a set Γ of conditional Σ-equations, we say that an unconditional equation $(\forall \tilde{X})\ t = t'$ is derivable from Γ iff it can be obtained by finite application of the following rules:

1. **Reflexivity**. For each variable declaration $\tilde{X}$ with $t \in \mathcal{T}_\Sigma(\tilde{X})$,

$$\frac{}{(\forall \tilde{X})\ t = t}$$

2. **Symmetry**.

$$\frac{(\forall \tilde{X})\ t = t'}{(\forall \tilde{X})\ t' = t}$$

3. **Transitivity**.

$$\frac{(\forall \tilde{X})\ t = t' \qquad (\forall \tilde{X})\ t' = t''}{(\forall \tilde{X})\ t = t''}$$

4. **Congruence**. For $f : s_1 \ldots s_n \to s$, $f : s'_1 \ldots s'_n \to s'$ in Σ with $s_1 \ldots s_n s \equiv_\leq s'_1 \ldots s'_n s'$, and for $t_i \in \mathcal{T}_\Sigma(\tilde{X})_{s_i}$, $t'_i \in \mathcal{T}_\Sigma(\tilde{X})_{s'_i}$, $1 \leq i \leq n$,

$$\frac{(\forall \tilde{X})\ t_1 = t'_1 \quad \ldots \quad (\forall \tilde{X})\ t_n = t'_n}{(\forall \tilde{X})\ f(t_1, \ldots, t_n) = f(t'_1, \ldots, t'_n)}$$

5. **Modus ponens**. For variable declarations $\tilde{X}, \tilde{Y}$, an assignment $\theta : \tilde{X} \to \mathcal{T}_\Sigma(\tilde{Y})$, and a conditional equation in Γ

$$(\forall \tilde{X})\ t = t' \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n$$

$$\frac{(\forall \tilde{Y})\ \overline{\theta}(u_1) = \overline{\theta}(v_1) \quad \ldots \quad (\forall \tilde{Y})\ \overline{\theta}(u_n) = \overline{\theta}(v_n)}{(\forall \tilde{Y})\ \overline{\theta}(t) = \overline{\theta}(t')}$$

Theorem 24. *Given an order-sorted theory (Σ, Γ), the above rules of deduction are sound and complete for deriving all the equations that are logical consequences of the axioms Γ.*

A signature morphism $H : (S, \leq, \Sigma) \to (S', \leq', \Sigma')$ induces a forgetful functor

$$\mathcal{U}_H : \mathbf{OSAlg}_{\Sigma'} \to \mathbf{OSAlg}_{\Sigma} \ ,$$

with $\mathcal{U}_H(A')_s = A'_{H(s)}$ and $\mathcal{U}_H(A')_f = A'_{H(f)}$ for each algebra $A' \in \mathbf{OSAlg}_{\Sigma'}$.

Definition 25. Given two order-sorted theories (Σ, Γ) and (Σ', Γ'), a *theory morphism* $H : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ is a signature morphism $H : \Sigma \to \Sigma'$ such that

$$\mathcal{U}_H(\mathbf{OSAlg}_{\Sigma', \Gamma'}) \subseteq \mathbf{OSAlg}_{\Sigma, \Gamma} \ ,$$

so that $\mathcal{U}_H$ restricts to a forgetful functor $\mathcal{U}_H : \mathbf{OSAlg}_{\Sigma', \Gamma'} \to \mathbf{OSAlg}_{\Sigma, \Gamma}$.

The institution $OSEqtl^{\mathrm{R}}$ is *liberal*, that is, for any theory morphism $H : (S, \leq, \Sigma, \Gamma) \to (S, \leq', \Sigma', \Gamma')$ we have an adjunction

$$\mathbf{OSAlg}_{\Sigma, \Gamma} \underset{\mathcal{U}_H}{\overset{\mathcal{F}_H}{\underset{\longleftarrow}{\overset{\dashrightarrow}{\bot}}}} \mathbf{OSAlg}_{\Sigma', \Gamma'}$$

12 Extending Order-Sorted Algebra to Membership Algebra

We can define an extension of institutions $(\Phi, \alpha, \beta) : OSEqtl^{\mathrm{R}} \rightarrowtail MEqtl$ as follows.

The functor Φ maps each order-sorted signature $\Sigma = (S, \leq, \Sigma)$ to the membership equational theory with signature $\Omega = (\hat{S}, \hat{\Sigma}, \pi : S \twoheadrightarrow \hat{S})$, where $\pi : S \twoheadrightarrow \hat{S}$ is the projection into $\equiv_{\leq}$-equivalence classes, and axioms

1. $(\forall x)\ x : s' \Leftarrow x : s \quad$ for each $s \leq s'$ in $(S, \leq)$,
2. $(\forall x_1 \ldots x_n)\ f(x_1, \ldots, x_n) : s \Leftarrow x_1 : s_1 \wedge \ldots \wedge x_n : s_n$
   for each $f : s_1 \ldots s_n \to s$ in Σ.

The natural transformation α maps each order-sorted conditional equation

$$(\forall \tilde{X})\ t = t' \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n$$

to the sentence

$$(\forall x_1 \ldots x_m)\ t = t' \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n \wedge \tilde{X}$$

where, by abuse of language, if $\tilde{X} = x_1 : s_1^1, \ldots, s_{l_1}^1; \ldots; x_m : s_1^m, \ldots, s_{l_m}^m$ then in the above sentence $\tilde{X}$ denotes the conjunction

$$x_1 : s_1^1 \wedge \ldots \wedge x_1 : s_{l_1}^1 \wedge \ldots \wedge x_m : s_1^m \wedge \ldots \wedge x_m : s_{l_m}^m \ .$$

In general, Φ maps an order-sorted theory (Σ, Γ) to the theory obtained by adding to $\Phi(\Sigma)$ the axioms $\alpha(\Gamma)$; and maps an order-sorted theory morphism

$H : (\Sigma, \Gamma) \longrightarrow (\Sigma', \Gamma')$ to the theory morphism with many-kinded signature morphism the induced $\hat{H} : \hat{\Sigma} \longrightarrow \hat{\Sigma}'$, and with sort map $H : S \longrightarrow S'$.

The natural transformation β is given by the functor

$$(\_)^\circ : \mathbf{Alg}_{\Phi(\Sigma,\Gamma)} \to \mathbf{OSAlg}_{\Sigma,\Gamma}$$

sending each $A \in \mathbf{Alg}_{\Phi(\Sigma,\Gamma)}$ to the order-sorted algebra with carrier sets $\{A_s\}_{s \in S}$ that satisfies $A_s \subseteq A_{s'}$ whenever $s \leq s'$, and has operations $A_{f:s_1 \ldots s_n \to s} : A_{s_1} \times \ldots \times A_{s_n} \to A_s$ thanks to the axioms in $\Phi(\Sigma)$, and that satisfies the axioms Γ thanks to the axioms $\alpha(\Gamma)$.

Theorem 26. *The functor* $(\_)^\circ : \mathbf{Alg}_{\Phi(\Sigma,\Gamma)} \to \mathbf{OSAlg}_{\Sigma,\Gamma}$ *has a left adjoint that is a full coreflection*

$$(\_)^\bullet : \mathbf{OSAlg}_{\Sigma,\Gamma} \to \mathbf{Alg}_{\Phi(\Sigma,\Gamma)} \ ,$$

where, for each $A \in \mathbf{OSAlg}_{\Sigma,\Gamma}$, *we generate* $A^\bullet$ *as the smallest* $\Phi(\Sigma,\Gamma)$*-algebra with* $A_k \subseteq A^\bullet_k$ *for each* $k \in \hat{S}$, $A^\bullet_s = A_s$ *for each* $s \in S$, *and defining for each* $f : k_1 \ldots k_n \to k$ *and* $(a_1, \ldots, a_n) \in A^\bullet_{k_1} \times \cdots \times A^\bullet_{k_n}$

$$A^\bullet_f(a_1, \ldots, a_n) = \begin{cases} A_f(a_1, \ldots, a_n) \in A_k & \text{if } (a_1, \ldots, a_n) \in dom(A_{f^{k_1 \ldots k_n}_k}) \\ \text{the term } f(a_1, \ldots, a_n) & \text{otherwise} \ . \end{cases}$$

This defines our desired extension map of institutions $(\Phi, \alpha, \beta) : \mathit{OSEqtl}^{\mathrm{R}} \rightarrowtail \mathit{MEqtl}$. This map is particularly nice, in that it automatically endows an order-sorted algebra A with *error elements* (the new terms that have a kind but not a sort) and allows a runtime error recovery discipline in the corresponding algebra $A^\bullet$ that is simpler and more satisfactory than that of retracts [16]. In a sense, it provides within the membership algebra framework the analogue of what was called an "error-supersort" recovery discipline in OBJ, but here error terms have a kind, not a sort, a feature that avoids many complications.

Corollary 27. *The extension map of institutions* $(\Phi, \alpha, \beta) : \mathit{OSEqtl}^{\mathrm{R}} \rightarrowtail \mathit{MEqtl}$ *is also a conservative map of logics [24] when we take into account the corresponding entailment systems of each of the logics as defined by the inference rules in Sections 11 and 4.*

Corollary 28. *Let* $H : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ *be a morphism of order-sorted theories. Then free constructions are preserved "in both directions" by restriction and extension, in the sense that, denoting by* $\mathcal{F}_H$ *the left adjoint to the forgetful functor*

$$\mathcal{U}_H : \mathbf{OSAlg}_{\Sigma',\Gamma'} \to \mathbf{OSAlg}_{\Sigma,\Gamma} \ ,$$

and by $\mathcal{F}_{\Phi(H)}$ *the left adjoint to the forgetful functor*

$$\mathcal{U}_{\Phi(H)} : \mathbf{Alg}_{\Phi(\Sigma',\Gamma')} \to \mathbf{Alg}_{\Phi(\Sigma,\Gamma)} \ ,$$

we have for each $A \in \mathbf{OSAlg}_{\Sigma,\Gamma}$

1. $\mathcal{F}_{\Phi(H)}(A^\bullet) \simeq (\mathcal{F}_H(A))^\bullet$
2. $\mathcal{F}_H(A) \simeq (\mathcal{F}_{\Phi(H)}(A^\bullet))^\circ$.

In particular, given a signature $\Sigma = (S, \leq, \Sigma)$, we can consider the signature $\Delta = (S, \leq, \emptyset)$ (whose models are $(S, \leq)$-sets) and the theory inclusion $(\Delta, \emptyset) \hookrightarrow (\Sigma, \Gamma)$, whose left adjoint is the free order-sorted algebra construction on an $(S, \leq)$-set of variables $\tilde{X}$, so that we get the following isomorphisms for the free and initial order-sorted algebras:

$$\mathcal{T}_{\Sigma,\Gamma}(\tilde{X}) \simeq (\mathcal{T}_{\Phi(\Sigma,\Gamma)}(\tilde{X}^\bullet))^\circ$$
$$\mathcal{T}_{\Sigma,\Gamma} \simeq \mathcal{T}^\circ_{\Phi(\Sigma,\Gamma)} \ .$$

Therefore, for all practical purposes we can perform all the usual order-sorted algebra constructions within the membership algebra framework.

13 Partial Membership Algebra

The partial analogue of membership algebra is *partial membership algebra*. It has a similar expressiveness to deal with sorts and subsorts and with both equations and sort predicates, and generalizes several well-known partial specification formalisms.

The notion of signature is slightly different from that for the total case. As in Example 1, we specify a partial order $(S, \leq)$ on sorts, with each $s \leq s'$ being syntactic sugar for an axiom $(\forall x)\ x : s' \Leftarrow x : s$. Furthermore, we require that each connected component k of $(S, \leq)$ should have a top element $\top_k$. This is because, without such a top, definedness of a term t in the kind k would then have a *disjunctive* expression by a family of membership assertions, instead of the single assertion $t : \top_k$. Kinds are then defined as connected components of the poset of sorts. As before, given a poset $S = (S, \leq)$, we denote by $\hat{S} = S/\equiv_\leq$ the set of its connected components. Now, we will call the connected components in $\hat{S}$ the *kinds* of the poset S, and will denote them either by k or by $[s]$.

Definition 29. A *signature* in partial membership equational logic is a triple $\Omega = (S, \leq, \Sigma)$ where $(S, \leq)$ is a poset such that each of its kinds k has a top element $\top_k$, and Σ is an $\hat{S}$-kinded signature. We call $(S, \leq)$ the poset of *sorts* of Ω.

To avoid unnecessary ambiguity we will assume that if $f : w \to k$ and $f : w' \to k'$ are operator declarations in Σ with w and w' words of the same length, then $w = w' \ \Leftrightarrow \ k = k'$.

Definition 30. Given a signature $\Omega = (S, \leq, \Sigma)$, a *partial Ω-algebra* A assigns:

1. to each $s \in S$ a set A_s, in such a way that whenever $s \leq s'$ then we have $A_s \subseteq A_{s'}$, and
2. to each $f : k_1 \dots k_n \to k$ in Σ, a *partial* function

$$A_f : A_{\top_{k_1}} \times \cdots \times A_{\top_{k_n}} \multimap A_{\top_k} \ .$$

Given two partial Ω-algebras A and B, an *Ω-homomorphism* $h : A \to B$ is an S-indexed family of (total) functions $\{h_s : A_s \to B_s\}_{s \in S}$ such that:

1. If $a \in A_s$ and $s \leq s'$, then $h_s(a) = h_{s'}(a)$.
2. For each $f : k_1 \dots k_n \to k$ in Σ, and $(a_1, \dots, a_n) \in A_{\top_{k_1}} \times \cdots \times A_{\top_{k_n}}$, if $A_f(a_1, \dots, a_n)$ is defined, then $B_f(h_{\top_{k_1}}(a_1), \dots, h_{\top_{k_n}}(a_n))$ is also defined and equal to $h_{\top_k}(A_f(a_1, \dots, a_n))$.

This determines a category $\mathbf{PAlg}_\Omega$.

Let $\Omega = (S, \leq, \Sigma)$ be a signature. Given a set of variables $X = \{x_1, \dots, x_m\}$, a *variable declaration* is a sequence

$$\tilde{X} = x_1 : \overline{s_1}; \dots; x_m : \overline{s_m}$$

where for each $i = 1, \dots, m$, $\overline{s_i}$ is a set of sorts $s^i_1, \dots, s^i_{l_i}$ having all the same kind.

Given a variable declaration $\tilde{X}$, by slight abuse of language we also denote by X the $\hat{S}$-indexed family of subsets of X associated to $\tilde{X}$ in the obvious way.

Atomic Ω-formulas are either equations of the form $t = t'$ where $t, t' \in T_\Sigma(X)_k$ for some $k \in \hat{S}$ (where $T_\Sigma(X)$ denotes as usual the free Σ-algebra on the variables X), or membership assertions of the form $t : s$, where $s \in S$ and $t \in T_\Sigma(X)_{[s]}$. General *Ω-sentences* are then Horn clauses of the form:

$$(i) \quad (\forall \tilde{X})\ t = t' \Leftarrow u_1 = v_1 \wedge \dots \wedge u_n = v_n \wedge w_1 : s_1 \wedge \dots \wedge w_m : s_m$$
$$(ii) \quad (\forall \tilde{X})\ t : s \Leftarrow u_1 = v_1 \wedge \dots \wedge u_n = v_n \wedge w_1 : s_1 \wedge \dots \wedge w_m : s_m$$

where t, t', u_i, v_i, and w_j are all terms in $T_\Sigma(X)$.

Given a partial Ω-algebra A and a variable declaration $\tilde{X}$, we can define assignments $a : \tilde{X} \to A$ in the obvious way (if $x : \overline{s}$ and s appears in $\overline{s}$, then we must have $a(x) \in A_s$), and then we can define by induction an $\hat{S}$-indexed family of partial functions $\overline{a} : T_\Sigma(X) \multimap A_\top$, where $A_\top = \{A_{\top_k}\}_{k \in \hat{S}}$, extending a in the usual way.

For atomic sentences we then define satisfaction by $A, a \models_\Omega t = t'$ meaning that $\overline{a}(t)$ and $\overline{a}(t')$ are both defined and $\overline{a}(t) = \overline{a}(t')$ (that is, we take an *existence equation* interpretation), and by $A, a \models_\Omega t : s$ meaning that $\overline{a}(t)$ is defined and $\overline{a}(t) \in A_s$. Satisfaction of Horn sentences is then defined in the obvious way.

Given a set Γ of Ω-sentences, we define $\mathbf{PAlg}_{\Omega,\Gamma}$ as the full subcategory of $\mathbf{PAlg}_\Omega$ determined by those partial Ω-algebras that satisfy all the sentences in Γ. In other words, the pair $T = (\Omega, \Gamma)$ is a *theory*, and $\mathbf{PAlg}_T = \mathbf{PAlg}_{\Omega,\Gamma}$ is the category of its models.

Example 2. The theory of categories T_{Cat} is a theory in partial membership equational logic. Its poset of sorts has sorts *Object* and *Arrow*, with *Object* $\leq$ *Arrow*; therefore, there is only one kind. The one-kinded signature has two unary

domain and codomain operations d, c, and a binary composition operation $\_;\_$. The theory T_{Cat} is presented by the following axioms:

$$
\begin{aligned}
(\forall f : \mathit{Arrow})\ & d(f) : \mathit{Object} \\
(\forall f : \mathit{Arrow})\ & c(f) : \mathit{Object} \\
(\forall a : \mathit{Object})\ & d(a) = a \\
(\forall a : \mathit{Object})\ & c(a) = a \\
(\forall f, g : \mathit{Arrow})\ & f; g : \mathit{Arrow} \Leftarrow c(f) = d(g) \\
(\forall f, g : \mathit{Arrow})\ & c(f) = d(g) \Leftarrow f; g : \mathit{Arrow} \\
(\forall f, g : \mathit{Arrow})\ & d(f; g) = d(f) \Leftarrow c(f) = d(g) \\
(\forall f, g : \mathit{Arrow})\ & c(f; g) = c(g) \Leftarrow c(f) = d(g) \\
(\forall a : \mathit{Object}, f : \mathit{Arrow})\ & f; a = f \Leftarrow c(f) = a \\
(\forall a : \mathit{Object}, f : \mathit{Arrow})\ & a; f = f \Leftarrow d(f) = a \\
(\forall f, g, h : \mathit{Arrow})\ & (f; g); h = f; (g; h) \Leftarrow c(f) = d(g) \wedge c(g) = d(h)
\end{aligned}
$$

It is easy to check that a model of T_{Cat} is exactly a category (in which objects coincide with identity arrows), and that a T_{Cat}-homomorphism is exactly a functor. Note that this example has some similarities with Example 1 (paths in a graph). A key difference is of course that the models in Example 1 are total, with error terms corresponding to undefined elements, where the models here are partial. However, one could easily define a partial variant `PATH`$_p$`[G :: GRAPH]` of the path parameterized module. The obvious theory morphism $T_{Cat} \longrightarrow$ `PATH`$_p$`[G :: GRAPH]` would then state that each free path category is indeed a category.

13.1 Partial Membership Algebra and Finitely Locally Presentable Categories

The expressive power of partial membership equational logic exactly corresponds to a logic whose categories of models are, up to equivalence of categories, the finitely locally presentable categories [12, 1].

Theorem 31. *Let (Ω, Γ) be a theory in partial membership equational logic. Then, $\mathbf{PAlg}_{\Omega,\Gamma}$ is a finitely locally presentable category.*

Proof sketch. First note that $\mathbf{PAlg}_{\Omega,\Gamma}$ is a full subcategory of $\mathbf{PAlg}_{\Omega}$.

Since (i) all categories of models definable in many-sorted Horn logic with equality are finitely locally presentable [1, pages 209–210 and Theorem 1.39], and (ii) a full reflective subcategory of a finitely locally presentable category that is closed under directed colimits is also finitely locally presentable [1, Theorem 1.39], it suffices to show:

1. $\mathbf{PAlg}_{\Omega}$ is definable in many-sorted Horn logic with equality, and
2. $\mathbf{PAlg}_{\Omega,\Gamma}$ is a full reflective subcategory of $\mathbf{PAlg}_{\Omega}$ closed under directed colimits.

To see 1 it is enough to check that, for $\Omega = (S, \leq, \Sigma)$, $\mathbf{PAlg}_\Omega$ is equivalent to the category of models of the many-kinded Horn theory with kinds $\hat{S}$, relation symbols

- for each $f : k_1 \ldots k_n \to k$ in Σ, a symbol $G(f) : k_1 \ldots k_n k$,
- for each sort $s \in S$ a unary predicate symbol $\_ : s$ of kind $[s]$,

and axioms

$$
\begin{array}{lll}
(i) & (\forall \overline{x}, y, y') \;\; y = y' \Leftarrow G(f)(\overline{x}, y) \wedge G(f)(\overline{x}, y'), & \text{for each } f \in \Sigma \\
(ii) & (\forall x) \;\; x : s' \Leftarrow x : s, & \text{for each } s \leq s' \text{ in } S \\
(iii) & (\forall x : k) \;\; x : \top_k \;\; . &
\end{array}
$$

The idea is that we can regard each partial Ω-algebra as a relational structure by viewing each partial operation f as a relation $G(f)$ satisfying (i), and can express the subsort structure by the axioms $(ii) - (iii)$ in such a way that Ω-homomorphisms exactly correspond to homomorphisms of the corresponding relational structure.

To see 2, by [1, Theorem 2.48], it is enough to show that $\mathbf{PAlg}_{\Omega,\Gamma}$ is closed in $\mathbf{PAlg}_\Omega$ under limits and directed colimits. Since the sorts of such limits and colimits are the set-theoretic limits and colimits of the corresponding diagrams of sets for each sort, and the operations are then induced in a universal way, it is somewhat tedious but not hard to check this property. □

Theorem 32. *Any finitely locally presentable category is equivalent to one of the form* $\mathbf{PAlg}_{\Omega,\Gamma}$ *for some theory* (Ω, Γ) *in partial membership equational logic.*

Proof sketch. It is enough to show that any finitely locally presentable category is equivalent to one of the form $\mathbf{PAlg}_{\Omega,\Gamma}$ in the special case in which (Ω, Γ) is an *essentially algebraic theory*, which, after a simple translation, is nothing but a theory in partial membership equational logic satisfying additional restrictions. Adámek and Rosický have already shown in [1, Theorem 3.36] that, indeed, any finitely locally presentable category is, up to equivalence, the category of algebras of an essentially algebraic theory.

Their notion of essentially algebraic theory is a special case of Reichel's *hierarchical equationally partial (hep)* theories [32] which can be regarded as a more general formulation of essentially algebraic theories as originally envisioned by Lawvere and Freyd [11]. Both the general hep notion and the more restricted one proposed by Adámek and Rosický have straightforward translations into partial membership equational logic. In fact, Reichel [32] makes explicit the embedding *HEP* $\hookrightarrow$ *PMSEqtl* into partial many-sorted equational logic with (conditional) existence equations, which is precisely the sublogic of *PMEqtl* obtained by requiring the poset of sorts $(S, \leq)$ to be discrete and restricting the sentences to Horn clauses involving only equations.

We give below the translation for the Adámek-Rosický formulation, in which a theory consists of a many-sorted signature Σ containing a subsignature Σ_t of total functions together with, for each $f \in \Sigma - \Sigma_t$, $f : w \to s$, a finite set $Def(f)$

of Σ_t-equations in the standard variables associated to the arity w, and finally a set E of Σ-equations.

An algebra for this theory is a partial many-sorted Σ-algebra A such that the operations in Σ_t are total, it satisfies the equations E as existence equations, and such that for each $f : w \to s$ in $\Sigma - \Sigma_t, A_f$ is defined on $(a_1, \ldots, a_n) \in A^w$ iff $A, (a_1, \ldots, a_n) \models Def(f)$.

This is of course exactly an algebra in membership equational logic with discrete poset of sorts those of Σ, operations Σ, and axioms E together with the axioms

- $f(\overline{x}) = f(\overline{x})$, for each $f \in \Sigma_t$,
- $f(\overline{x}) = f(\overline{x}) \iff e_1 \wedge \ldots \wedge e_n$ for each $f \in \Sigma - \Sigma_t$ with $Def(f) = \{e_1, \ldots, e_n\}$. □

13.2 The Institution of Partial Membership Algebra

Definition 33. Given two signatures $\Omega = (S, \leq, \Sigma)$ and $\Omega' = (S', \leq', \Sigma')$ in partial membership equational logic, a *signature morphism* $H : \Omega \to \Omega'$ is given by

1. a monotonic function $H_1 : (S, \leq) \to (S', \leq')$, and
2. an $\hat{S}^* \times \hat{S}$-indexed family of functions

$$H_2 = \{H_{2,w,k} : \Sigma_{w,k} \to \Sigma'_{\hat{H}_1(w), \hat{H}_1(k)}\}_{(w,k) \in \hat{S}^* \times \hat{S}}$$

where $\hat{H}_1$ is the induced function on connected components extended also homomorphically to strings. By abuse of notation, we will drop the subscripts in H_1, H_2 and write H throughout.

Such a signature morphism induces a forgetful functor

$$\mathcal{U}^p_H : \mathbf{PAlg}_{\Omega'} \to \mathbf{PAlg}_{\Omega} \ ,$$

where for each algebra $A' \in \mathbf{PAlg}_{\Omega'}$ we have

- $\mathcal{U}^p_H(A')_s = A'_{H(s)}$ for every $s \in S$,
- for each $f : k_1 \ldots k_n \to k$ in Σ,

$$\mathcal{U}^p_H(A')_f = A'_{H_{k_1 \ldots k_n, k}(f)} \cap (A'_{H(\top_{k_1})} \times \cdots \times A'_{H(\top_{k_n})} \times A'_{H(\top_k)}) \ .$$

Definition 34. Given two theories (Ω, Γ) and (Ω', Γ'), a *theory morphism* $H : (\Omega, \Gamma) \to (\Omega', \Gamma')$ is a signature morphism $H : \Omega \to \Omega'$ such that

$$\mathcal{U}^p_H(\mathbf{PAlg}_{\Omega', \Gamma'}) \subseteq \mathbf{PAlg}_{\Omega, \Gamma} \ ,$$

so that $\mathcal{U}^p_H$ restricts to a forgetful functor $\mathcal{U}^p_H : \mathbf{PAlg}_{\Omega', \Gamma'} \to \mathbf{PAlg}_{\Omega, \Gamma}$.

The above definitions specify the category of signatures $\mathbf{Sign}_p$, and the model functor $\mathbf{PAlg} : \mathbf{Sign}_p^{op} \to \mathbf{Cat}$. We now need to define sentences. However, because of the useful generality of allowing signature morphisms with monotonic $H : (S, \leq) \to (S', \leq')$ that may not preserve the tops[13] of connected components, the definition of the sentence functor $sen_p : \mathbf{Sign}_p \to \mathbf{Set}$ is somewhat subtle. The problem is that the meaning of a sentence can be drastically altered by a small change in its "ambient" signature. Consider for example a signature Ω with one sort s and with a unary operation f and a binary operation g, and let $\Omega' \supseteq \Omega$ be the signature extending Ω by adding a new sort s' with $s \leq s'$. Then, the meaning of the Ω-sentence

$$(\dagger) \qquad (\forall x : s)\ f(x) = g(x, x)$$

is destroyed if it is mapped to itself by $sen(J)$, for $J : \Omega \hookrightarrow \Omega'$ the inclusion of signatures. This is because an Ω'-algebra A' can satisfy the sentence $(\dagger)$ while $\mathcal{U}_J^p(A')$ may not, since in A' for an assignment $a : \{x\} \to A'_s$ we can have $\overline{a}(f(x)), \overline{a}(g(x, x)) \in A'_{s'} - A'_s$. The proper translation of $(\dagger)$ as an Ω'-sentence is

$$(\forall x : s)\ f(x) = g(x, x) \wedge f(x) : s \wedge g(x, x) : s \ .$$

To deal with this problem we introduce the notion of *Ω-envelope* of a sentence. This is a semantically equivalent sentence that, when translated, captures the intended meaning.

Definition 35. Let Ω be a signature in partial membership equational logic. Then, for an equation $t = t'$ we define

$$env_\Omega(t = t') \ = \ t = t' \wedge \bigwedge_{u \in NVST(\{t,t'\})} u : \top_{k(u)} \ ,$$

where $NVST(\{t, t'\})$ is the set of nonvariable subterms of either t or t', and where for each such subterm u, $k(u)$ denotes its corresponding kind.

Similarly, for a membership assertion $t : s$ we define

$$env_\Omega(t : s) \ = \ t : s \wedge \bigwedge_{u \in NVST(t) - \{t\}} u : \top_{k(u)} \ .$$

Then, for a Horn sentence φ

$$(\forall \tilde{X})\ at \Leftarrow u_1 = v_1 \wedge \ldots \wedge u_n = v_n \wedge w_1 : s_1 \wedge \ldots \wedge w_m : s_m$$

with at an equation or membership assertion, we define $env_\Omega(\varphi)$ as the sentence

$$\begin{aligned}(\forall \tilde{X})\ env_\Omega(at) \Leftarrow\ & env_\Omega(u_1 = v_1) \wedge \ldots \wedge env_\Omega(u_n = v_n) \\ & \wedge\ env_\Omega(w_1 : s_1) \wedge \ldots \wedge env_\Omega(w_m : s_m)\end{aligned}$$

where now the consequent is in fact a conjunction.

[13] This generality is actually quite important in practice. Several examples of non-top-preserving theory morphisms can be found in [26], which provides a good case study of the usefulness and expressiveness of partial membership equational logic in the area of 2-categorical and double-categorical models for rewriting logic and tile logic, their relationships, and their concurrency applications.

Extending the satisfaction relation in the obvious way to universally quantified implications with conjunctions of atomic formulas in their antecedent and consequent parts, it is easy to check the following

Lemma 36. *For $A \in \mathbf{PAlg}_\Omega$ and φ a Horn sentence, $A \models_\Omega \varphi$ iff $A \models_\Omega env_\Omega(\varphi)$.*

This suggests extending the notion of sentence in *PMEqtl* from Horn clauses to universally quantified implications between conjunctions of atomic formulas. This of course does not change the classes of algebras that can be defined, since any such implication is equivalent to the set of Horn clauses obtained by implying each one of the atomic formulas in the consequent from the given set of premises, but will allow us to give a good description of the functor of sentences.

In fact, let $H : \Omega \to \Omega'$ be a signature morphism. Then, we have the following systematic relationship between sentences in Ω and sentences in Ω' with respect to satisfaction.

Lemma 37 (Satisfaction Lemma). *For $H : \Omega \to \Omega'$ a signature morphism, φ an implication between conjunctions of atomic formulas in the sentences of Ω, and $A' \in \mathbf{PAlg}_{\Omega'}$, we have*

$$\mathcal{U}^p_H(A') \models_\Omega \varphi \quad \Longleftrightarrow \quad A' \models_{\Omega'} H(env_\Omega(\varphi))$$

where H denotes also the obvious translation of Ω-terms into Ω'-terms and of S-sorted variables into S'-sorted variables induced by the signature morphism H.

Proof sketch. We deal with the case of atomic sentences; the extension to implications between conjunctions of atomic formulas is straightforward.

Consider an Ω-equation $(\forall \tilde{X})\ t = t'$. We have $\mathcal{U}^p_H(A') \models_\Omega (\forall \tilde{X})\ t = t'$ iff (by previous lemma) $\mathcal{U}^p_H(A') \models_\Omega (\forall \tilde{X})\ env_\Omega(t = t')$.

Defining $H(\tilde{X})$ as the Ω'-declaration such that $x : s_1 \ldots s_n$ is in $\tilde{X}$ iff $x : H(s_1) \ldots H(s_n)$ is in $H(\tilde{X})$ for each $x \in X$, and noticing that, by construction, $\mathcal{U}^p_H(A')_s = A'_{H(s)}$, it is easy to show by reasoning on the corresponding assignments that $\mathcal{U}^p_H(A') \models_\Omega (\forall \tilde{X})\ env_\Omega(t = t')$ implies $A' \models_{\Omega'} (\forall H(\tilde{X}))\ env_{\Omega'}(H(t) = H(t'))$.

In a similar way, for a membership assertion $(\forall \tilde{X})\ t : s$, we have that $\mathcal{U}^p_H(A') \models_\Omega (\forall \tilde{X})\ env_\Omega(t : s)$ implies $A' \models_{\Omega'} (\forall H(\tilde{X}))\ env_{\Omega'}(H(t) : H(s))$.

In this way we can show the implication

$$\mathcal{U}^p_H(A') \models_\Omega \varphi \quad \Rightarrow \quad A' \models_{\Omega'} H(env_\Omega(\varphi)) \ .$$

The key idea for the converse implication can be captured by considering φ an atomic sentence of the form $(\forall \tilde{X})\ t : s$. Then $\mathcal{U}^p_H(A') \models_\Omega \varphi$ means that for any assignment $a : \tilde{X} \to \mathcal{U}^p_H(A')$, $\overline{a}(t)$ is defined and $\overline{a}(t) \in \mathcal{U}^p_H(A')_s = A'_{H(s)}$. This of course implies that for each nonvariable subterm u of t, $\overline{a}(u)$ is defined and $\overline{a}(u) \in \mathcal{U}^p_H(A')_{\top_{k(u)}} = A'_{H(\top_{k(u)})}$. Since each assignment $a : \tilde{X} \to \mathcal{U}^p_H(A')$ can be regarded as an assignment $a' : H(\tilde{X}) \to A'$ and viceversa, this is equivalent

to requiring that for each $a' : H(\tilde{X}) \to A'$, $\overline{a'}(H(u))$ is defined and $\overline{a'}(H(u)) \in A'_{H(\top_{k(u)})}$, that is, $A' \models_{\Omega'} (\forall H(\tilde{X}))\ H(u) : H(\top_{k(u)})$. Therefore, we have the equivalence

$$\mathcal{U}_H^p(A') \models_\Omega (\forall \tilde{X})\ t : s \quad \Longleftrightarrow \quad A' \models_{\Omega'} (\forall H(\tilde{X}))\ H(env_\Omega(t : s))$$

as desired. □

The obvious candidate for $sen_p(H)$ would then be $sen_p(H)(\varphi) = H(env_\Omega(\varphi))$ but this definition requires some qualifications in order to be functorial. Think for example of the case $H = 1_\Omega$, where $sen_p(H)$ has to be the identity!

This functoriality problem can be resolved by giving to sentences a somewhat more subtle structure. Notice that $sen_p(\Omega)$ is the set of sentences of the form

$$(\forall \tilde{X})\ at_1 \wedge \ldots \wedge at_n \Leftarrow at'_1 \wedge \ldots \wedge at'_m$$

with the at_i, at'_j Ω-atoms, which can be viewed as an equationally defined abstract data type in which $\_ = \_$ is commutative, $\_ \Leftarrow \_$ is a constructor, $(\forall \tilde{X})\_$ is another constructor subjected to a membership condition (the variables in the atomic formulas must appear in X), and $\_ \wedge \_$ is an associative, commutative, and idempotent operator with neutral element *true*. The additional structure solving the functoriality problem is obtained by imposing the following additional equations on $sen_p(\Omega)$:

- $f(x_1, \ldots, x_i, \ldots, x_n) : s \wedge x_i : \top_{k_i} \ = \ f(x_1, \ldots, x_i, \ldots, x_n) : s$ for each $f : k_1 \ldots k_n \to k$ in Σ and $s \in k$,
- $(x : \top_k \wedge x = y) \ = \ (x = y)$ for all $k \in \hat{S}$,
- $(x : s \wedge x : s') \ = \ (x : s)$ for all $s \leq s'$ in S,

which of course always yield semantically equivalent sentences with respect to the satisfaction relation $\models_\Omega$. It is also easy to check that the above equations are Church-Rosser and terminating *modulo* the associativity, commutativity, and identity of the conjunction operator $\_ \wedge \_$. We then define $sen_p(\Omega)$ as the abstract data type obtained by imposing the above additional equations, which can be explicitly described by considering the *canonical forms* of expressions (modulo associativity, commutativity, and identity) that are obtained using the above equations as rewrite rules.

Lemma 38. *With $sen_p(\Omega)$ defined as above, $sen_p(H)(\varphi) = H(env_\Omega(\varphi))$ defines a functor $sen_p : \mathbf{Sign}_p \to \mathbf{Set}$.*

Proof. First of all, sen_p preserves identities, because, by repeated application of the first two equations, we can show $sen_p(1_\Omega)(\varphi) = env_\Omega(\varphi) = \varphi$ for any sentence $\varphi \in sen_p(\Omega)$.

To see that sen_p preserves composition, let $\Omega \stackrel{H}{\to} \Omega' \stackrel{G}{\to} \Omega''$ be signature morphisms. We have to show that $GH(env_\Omega(\varphi)) = G(env_{\Omega'}(H(env_\Omega(\varphi))))$ in

$sen_p(\Omega'')$. It is enough to check this for atomic sentences, since the extension to implications is quite obvious. We will do the case of an equation. We have

$$
\begin{array}{l}
G(env_{\Omega'}(H(env_{\Omega}(t = t')))) = \\
G(env_{\Omega'}(H(t = t' \wedge \bigwedge_{u \in NVST(\{t,t'\})} u : \top_{k(u)}))) = \\
G(env_{\Omega'}(H(t) = H(t') \wedge \bigwedge_{H(u) \in NVST(\{H(t),H(t')\})} H(u) : H(\top_{k(u)}))) = \\
G(H(t) = H(t') \wedge \bigwedge_{H(u) \in NVST(\{H(t),H(t')\})} H(u) : \top_{k'(H(u))} \wedge \\
\bigwedge_{H(u) \in NVST(\{H(t),H(t')\})} (H(u) : H(\top_{k(u)}) \wedge \bigwedge_{v \in NVST(H(u))} v : \top_{k'(v)})) = (*) \\
G(H(t) = H(t') \wedge \bigwedge_{H(u) \in NVST(\{H(t),H(t')\})} H(u) : H(\top_{k(u)})) = \\
GH(env_{\Omega}(t = t'))
\end{array}
$$

where the step $(*)$ is justified by the fact that $H(\top_{k(u)}) \leq \top_{k'(H(u))}$ and that each nonvariable subterm of each $H(u)$ is also a nonvariable subterm of either $H(t)$ or $H(t')$, plus repeated application of the third equation. □

This lemma and the Satisfaction Lemma 37 show that indeed *PMEqtl* is an institution. We now proceed to show that it is a *liberal* institution.

Theorem 39. *PMEqtl is a liberal institution.*

Proof sketch. We need to show that if $H : (\Omega, \Gamma) \to (\Omega', \Gamma')$ is a theory morphism, then the functor $\mathcal{U}^p_H : \mathbf{PAlg}_{\Omega',\Gamma'} \to \mathbf{PAlg}_{\Omega,\Gamma}$ has a left adjoint. Since we have already shown that $\mathbf{PAlg}_{\Omega',\Gamma'}$ and $\mathbf{PAlg}_{\Omega,\Gamma}$ are finitely locally presentable categories in Theorem 31, by the Adjoint Functor Theorem for locally presentable categories [1, Theorem 1.66], it is enough to show that $\mathcal{U}^p_H$ preserves limits and directed colimits. This is somewhat tedious but not very hard to check, using the explicit set-theoretic descriptions of such limits and colimits. □

The above theorem relies on quite abstract principles and constructions, and does not provide an explicit, proof-theoretic, construction of the left adjoint. However, thanks to the general results about extension maps of liberal institutions in Section 10 (see Theorem 17) and our explicit construction of the left adjoint to the functor $\mathcal{U}_H$ for H a theory morphism in *MEqtl* given in Section 7 (see Theorem 10), we will derive in the next section a very explicit construction of the left adjoints to forgetful functors $\mathcal{U}^p_H$ for the above case of theory morphisms in *PMEqtl*.

14 Extending Partial Membership Algebra to Membership Algebra

We are now ready to define an extension map of institutions. Using Proposition 14, the corresponding generalized map of institutions can be characterized by a triple

$$(\Psi_0, \gamma, (\_)^\circ) : PMEqtl \rightarrowtail MEqtl \ .$$

The functor Ψ_0 sends signatures in *PMEqtl* to theories in *MEqtl*. A signature $\Omega = (S, \leq, \Sigma)$ is mapped to the theory $\Psi_0(\Omega)$ with signature $(\hat{S}, \hat{\Sigma}, \pi : S \twoheadrightarrow \hat{S})$, where $\pi : S \twoheadrightarrow \hat{S}$ is the projection into equivalence classes, and with axioms $(\forall x)\ x : s' \Leftarrow x : s$ for each $s \leq s'$ in $(S, \leq)$.

Signature morphisms are mapped in the obvious way, namely $\Psi_0(H_1, H_2) = (\hat{H}_1, H_1, H_2)$.

Therefore, there is a functor $\Psi^\diamond : \mathbf{Sign}_p \to \mathbf{Sign}$ such that $\Psi^\diamond = sign \circ \Psi_0$.

The family of functions

$$\{\gamma_\Omega : sen_p(\Omega) \to \mathcal{P}(sen(\Psi^\diamond(\Omega)))\}_{\Omega \in \mathbf{Sign}_p}$$

maps each *canonical* form $(\forall \tilde{X})\ at_1 \wedge \ldots \wedge at_n \Leftarrow at'_1 \wedge \ldots \wedge at'_m$ in $sen_p(\Omega)$ to the set of sentences

$$\{(\forall X)\ at \Leftarrow env_\Omega(at'_1) \wedge \ldots \wedge env_\Omega(at'_m) \wedge \tilde{X} \mid at \text{ is an atomic formula in the conjunction } env_\Omega(at_1) \wedge \ldots \wedge env_\Omega(at_n)\}$$

where, by abuse of language, if $\tilde{X} = x_1 : s^1_1, \ldots, s^1_{l_1}; \ldots; x_m : s^m_1, \ldots, s^m_{l_m}$, then in the above sentences $\tilde{X}$ denotes the conjunction

$$x_1 : s^1_1 \wedge \ldots \wedge x_1 : s^1_{l_1} \wedge \ldots \wedge x_m : s^m_1 \wedge \ldots \wedge x_m : s^m_{l_m} \ .$$

The natural transformation $(\_)^\circ : \mathbf{Alg} \circ \Psi_0^{op} \Rightarrow \mathbf{PAlg}$ is the functor sending each algebra $A \in \mathbf{Alg}_{\Psi_0(\Omega)}$ to the partial Ω-algebra $A^\circ \in \mathbf{PAlg}_\Omega$ obtained by restricting each operation $A_f : A_{k_1} \times \cdots \times A_{k_n} \to A_k$ to the partial function

$$A^\circ_f = A_f \cap (A_{\top_{k_1}} \times \cdots \times A_{\top_{k_n}} \times A_{\top_k}) \ .$$

We first need to check that this defines a generalized map of institutions by means of Proposition 14.

For each $H : \Omega \to \Omega'$ in $\mathbf{Sign}_p$, denoting $\Psi_0(\Omega') = (\Psi^\diamond(\Omega'), Ax_{\Omega'})$, we need to show that for each $\varphi \in sen_p(\Omega)$ we have

$$\text{(ii)} \qquad \mathbf{Alg}_{\Psi^\diamond(\Omega'), Ax_{\Omega'} \cup \Psi^\diamond(H)(\gamma_\Omega(\varphi))} = \mathbf{Alg}_{\Psi^\diamond(\Omega'), Ax_{\Omega'} \cup \gamma_{\Omega'}(H(\varphi))}$$

where we have shortened $sen_p(H)$ to H and $sen(\Psi^\diamond(H))$ to $\Psi^\diamond(H)$.

That is, we have to show that for any algebra $A \in \mathbf{Alg}_{\Psi_0(\Omega')}$ we have

$$A \models_{\Psi^\diamond(\Omega')} \Psi^\diamond(H)(\gamma_\Omega(\varphi)) \iff A \models_{\Psi^\diamond(\Omega')} \gamma_{\Omega'}(H(\varphi)) \ .$$

We will prove it for φ an equation of the form $(\forall \tilde{X})\ t = t'$, and will leave the remaining cases to the interested reader. On the one hand,

$$\begin{aligned}
&\Psi^\diamond(H)(\gamma_\Omega((\forall \tilde{X})\ t = t')) = \\
&\Psi^\diamond(H)(\{(\forall X)\ t = t' \Leftarrow \tilde{X}\} \cup \{(\forall X)\ u : \top_{k(u)} \Leftarrow \tilde{X} \mid u \in NVST(\{t, t'\})\}) = \\
&\{(\forall \hat{H}(X))\ H(t) = H(t') \Leftarrow H(\tilde{X})\} \cup \\
&\qquad \{(\forall \hat{H}(X))\ H(u) : H(\top_{k(u)}) \Leftarrow H(\tilde{X}) \mid u \in NVST(\{t, t'\})\} \ .
\end{aligned}$$

On the other hand, assuming $H(\top_{k(u)}) \neq \top_{k'(H(u))}$ (the other case being easier),

$$\begin{array}{l}
\gamma_{\Omega'}(H((\forall \tilde{X})\ t = t')) = \\
\gamma_{\Omega'}((\forall H(\tilde{X}))\ H(t) = H(t') \wedge \bigwedge_{u \in NVST(\{t,t'\})} H(u) : H(\top_{k(u)})) = \\
\{(\forall \hat{H}(X))\ H(t) = H(t') \Leftarrow H(\tilde{X})\} \cup \\
\qquad \{(\forall \hat{H}(X))\ H(u) : \top_{k'(H(u))} \Leftarrow H(\tilde{X}) \mid u \in NVST(\{t,t'\})\} \cup \\
\qquad \bigcup_{u \in NVST(\{t,t'\})} (\{(\forall \hat{H}(X))\ H(u) : H(\top_{k(u)}) \Leftarrow H(\tilde{X})\} \cup \\
\qquad\qquad \{(\forall \hat{H}(X))\ v : \top_{k'(v)} \Leftarrow H(\tilde{X}) \mid v \in NVST(H(u)) - \{H(u)\}\})\ ,
\end{array}$$

which is clearly semantically equivalent to the previous set of sentences under the axioms $(\forall x)\ x : s' \Leftarrow x : s$ for each $s \leq s'$ in $(S, \leq)$.

We have also to prove that for each $\Omega \in \mathbf{Sign}_p, \varphi \in sen_p(\Omega)$, and $A \in \mathbf{Alg}_{\Psi_0(\Omega)}$ we have

$$\text{(iii)} \qquad A \models_{\Psi^\diamond(\Omega)} \gamma_\Omega(\varphi) \iff A^\circ \models_\Omega \varphi\ .$$

Again, let us treat the case $\varphi = (\forall \tilde{X})\ t = t'$, and let us leave the remaining cases to the interested reader. Then, $\gamma_\Omega((\forall \tilde{X})\ t = t')$ is the set of axioms

$$\{(\forall X)\ t = t' \Leftarrow \tilde{X}\} \cup \{(\forall X)\ u : \top_{k(u)} \Leftarrow \tilde{X} \mid u \in NVST(\{t,t'\})\}\ .$$

First note that an assignment $a : X \to A$ satisfying $\tilde{X}$ is exactly an assignment $a : \tilde{X} \to A^\circ$. Then note that $\overline{a}(t)$ and $\overline{a}(t')$ are defined in A° for such an assignment iff

$$A, a \models_{\Psi^\diamond(\Omega)} (\forall X)\ u : \top_{k(u)} \Leftarrow \tilde{X}$$

for each $u \in NVST(\{t,t'\})$, because A satisfies the axioms $(\forall x)\ x : s' \Leftarrow x : s$ for each $s \leq s'$ in $(S, \leq)$.

And, assuming that both sides are defined in A°, then

$$A^\circ, a \models_\Omega (\forall \tilde{X})\ t = t' \quad \text{iff} \quad A, a \models_{\Psi^\diamond(\Omega)} (\forall X)\ t = t' \Leftarrow \tilde{X}\ .$$

Finally, as in Proposition 14, Ψ_0 is extended to a signature-preserving functor $\Psi : \mathbf{Th}_p \to \mathbf{Th}$ sending each theory (Ω, Γ) to the theory $(\Psi^\diamond(\Omega), Ax_\Omega \cup \bigcup \gamma_\Omega(\Gamma))$.

In order to prove that this generalized map of institutions is an extension map, we have to show now that for any $(\Omega, \Gamma) \in \mathbf{Th}_p$ the forgetful functor

$$(\_)^\circ : \mathbf{Alg}_{\Psi^\diamond(\Omega), Ax_\Omega \cup \bigcup \gamma_\Omega(\Gamma)} \to \mathbf{PAlg}_{\Omega,\Gamma}\ ,$$

which is just the restriction of the forgetful functor

$$(\_)^\circ : \mathbf{Alg}_{\Psi^\diamond(\Omega), Ax_\Omega} \to \mathbf{PAlg}_\Omega\ ,$$

has a left adjoint

$$(\_)^\bullet : \mathbf{PAlg}_{\Omega,\Gamma} \to \mathbf{Alg}_{\Psi^\diamond(\Omega), Ax_\Omega \cup \bigcup \gamma_\Omega(\Gamma)}$$

such that the unit of the adjunction is an isomorphism.

But, since the restriction of $(\_)^\circ$ is a restriction to full subcategories closed under isomorphism, if we show that the original functor has a left adjoint $(\_)^\bullet$ whose unit is an isomorphism, and also show that the image under $(\_)^\bullet$ of $\mathbf{PAlg}_{\Omega,\Gamma}$ is included in $\mathbf{Alg}_{\Psi(\Omega,\Gamma)}$, that is, that $(\_)^\bullet$ also has a restriction, the result will clearly hold for any $(\Omega,\Gamma) \in \mathbf{Th}_p$ by taking such restriction.

Indeed, given $A \in \mathbf{PAlg}_\Omega$, define $A^\bullet \in \mathbf{Alg}_{\Psi_0(\Omega)}$ as the smallest algebra with $A_k \subseteq A^\bullet_k$ for each $k \in \hat{S}$, $A^\bullet_s = A_s$ for each $s \in S$, and defining for each $f : k_1 \dots k_n \to k$ and $(a_1, \dots, a_n) \in A^\bullet_{k_1} \times \cdots \times A^\bullet_{k_n}$

$$A^\bullet_f(a_1,\dots,a_n) = \begin{cases} A_f(a_1,\dots,a_n) \in A_k & \text{if } (a_1,\dots,a_n) \in dom(A_f) \\ \text{the } \textit{term } f(a_1,\dots,a_n) & \text{otherwise .} \end{cases}$$

Since the subsort structure is left unchanged, $A^\bullet$ clearly satisfies the subsort axioms in $\Psi_0(\Omega)$. Also, by construction we have $A = (A^\bullet)^\circ$ for any $A \in \mathbf{PAlg}_\Omega$. Taking such an identity map as the unit map, and assuming an algebra $B \in \mathbf{Alg}_{\Psi_0(\Omega)}$ and an Ω-homomorphism $h : A \to B^\circ$, we can inductively check that there is a unique $\Psi_0(\Omega)$-homomorphism $\overline{h} : A^\bullet \to B$ such that $\overline{h}^\circ = h$. The base case is $A \subseteq A^\bullet$ for which $\overline{h}$ must coincide with h. Then, reasoning inductively on the depth of terms in $A^\bullet$, if $\overline{h}$ is uniquely defined for terms of depth $\leq n$ and $f(a_1,\dots,a_n)$ is a term of depth $n+1$, we must necessarily have $\overline{h}(f(a_1,\dots,a_n)) = B_f(\overline{h}(a_1),\dots,\overline{h}(a_n))$, which uniquely defines $\overline{h}$ for terms of depth $n+1$ and ensures the homomorphic property.

We finally have to check that if $A \in \mathbf{PAlg}_{\Omega,\Gamma}$, then $A^\bullet \in \mathbf{Alg}_{\Psi(\Omega,\Gamma)}$. Since $A^\bullet$ clearly satisfies each of the axioms in $\Psi_0(\Omega)$, what we need to show is that if

$$(\forall \tilde{X})\ at \Leftarrow at_1 \wedge \ldots \wedge at_n$$

is an axiom in Γ (it is enough to consider Horn clauses), then for each atomic formula at' in $env_\Omega(at)$ we have

$$A^\bullet \models (\forall X)\ at' \Leftarrow env_\Omega(at_1) \wedge \ldots \wedge env_\Omega(at_n) \wedge \tilde{X}\ ,$$

which follows trivially from the already checked condition (ii) on $(\Psi_0, \gamma, (\_)^\circ)$ being a generalized map of institutions, since $(A^\bullet)^\circ = A$ and $A \models_\Omega (\forall \tilde{X})\ at \Leftarrow at_1 \wedge \ldots \wedge at_n$ by hypothesis.

Thus, we have proved

Theorem 40. *$(\Psi, \gamma, (\_)^\circ) : PMEqtl \rightarrowtail MEqtl$ is an extension map of institutions.*

From this theorem, using the results in Section 10, we can derive the following useful corollaries.

Corollary 41. *We can "borrow" the entailment system of MEqtl to endow the institution PMEqtl with the following entailment system: Given a theory $(\Omega,\Gamma) \in \mathbf{Th}_p$ and a sentence $\varphi \in sen_p(\Omega)$, we define*

$$\Gamma \vdash^p_\Omega \varphi \iff Ax_\Omega \cup \bigcup \gamma_\Omega(\Gamma) \vdash_{\Psi^\circ(\Omega)} \psi \quad \textit{for all } \psi \in \gamma_\Omega(\varphi)\ .$$

This entailment system makes PMEqtl into a complete *logic, and makes the extension map PMEqtl $\rightarrowtail$ MEqtl into a* conservative *generalized map of logics.*

Corollary 42. *A signature morphism $H : \Omega \to \Omega'$ is a theory morphism $H : (\Omega, \Gamma) \to (\Omega', \Gamma')$ in $\mathbf{Th}_p$ iff for each $\varphi \in \Gamma$ we have $\Gamma' \vdash^p_{\Omega'} \varphi$.*

Corollary 43. *Let $H : (\Omega, \Gamma) \to (\Omega', \Gamma')$ be a theory morphism in $\mathbf{Th}_p$. Then free constructions are preserved "in both directions" by restriction and extension, in the sense that, denoting by $\mathcal{F}^p_H$ the left adjoint to the forgetful functor*

$$\mathcal{U}^p_H : \mathbf{PAlg}_{\Omega',\Gamma'} \to \mathbf{PAlg}_{\Omega,\Gamma} \ ,$$

and by $\mathcal{F}_{\Psi(H)}$ the left adjoint to the forgetful functor

$$\mathcal{U}_{\Psi(H)} : \mathbf{Alg}_{\Psi(\Omega',\Gamma')} \to \mathbf{Alg}_{\Psi(\Omega,\Gamma)} \ ,$$

we have for each $A \in \mathbf{PAlg}_{\Omega,\Gamma}$

1. $\mathcal{F}_{\Psi(H)}(A^\bullet) \simeq (\mathcal{F}^p_H(A))^\bullet$
2. $\mathcal{F}^p_H(A) \simeq (\mathcal{F}_{\Psi(H)}(A^\bullet))^\circ$.

In particular, given a signature $\Omega = (S, \leq, \Sigma)$, we can consider the signature $\Delta = (S, \leq, \emptyset)$ (whose models are $(S, \leq)$-sets) and the theory inclusion $(\Delta, \emptyset) \hookrightarrow (\Omega, \Gamma)$, whose left adjoint is the free algebra construction on an $(S, \leq)$-set of variables $\tilde{X}$, so that we get the following explicit construction of the free $(\mathcal{T}^p_{\Omega,\Gamma}(\tilde{X}))$ and initial $(\mathcal{T}^p_{\Omega,\Gamma})$ algebras in $\mathbf{PAlg}_{\Omega,\Gamma}$:

$$\mathcal{T}^p_{\Omega,\Gamma}(\tilde{X}) = (\mathcal{T}_{\Psi(\Omega,\Gamma)}(\tilde{X}^\bullet))^\circ$$
$$\mathcal{T}^p_{\Omega,\Gamma} = \mathcal{T}^\circ_{\Psi(\Omega,\Gamma)} \ .$$

Therefore, both in terms of logical deduction and of free, relatively free, and initial algebras, we can perform all our constructions and reasoning for partial membership algebra within the logical framework of total membership algebra.

15 Membership Algebra as a Logical Framework for Equational Specification

Partial membership equational logic is a quite expressive formalism for specifying partial algebras, in a way entirely analogous to how membership equational logic is a flexible specification formalism. In particular, many other equational or Horn logics for partial specification can be mapped by embeddings or by weak embeddings into *PMEqtl*.

We have already discussed the embeddings

$$\mathit{EssAlg}^{\mathrm{AR}} \hookrightarrow \mathit{HEP} \hookrightarrow \mathit{PMSEqtl} \hookrightarrow \mathit{PMEqtl}$$

in the proof of Theorem 32, where the essentially algebraic theories in the sense of Adámek and Rosický [1] are a special case of hierarchical equationally partial theories in the sense of Reichel [32], and can then be naturally expressed

as theories in partial many-sorted equational logic with (conditional) existence equations that, in turn, is a sublogic of partial membership equational logic, namely, the sublogic obtained by requiring that the posets of sorts are discrete and by allowing only equations as atoms.

We can in addition connect those embeddings with "Mossakowski's web" [27] of embeddings and weak embeddings of institutions showing the essential equivalence between many different formalisms for partial specification. Mossakowski has shown that the categories of models defined by each of the logics in his web are all the same, namely, the finitely locally presentable categories. We refer to [27] for the details of this web and briefly summarize below his notation for the different additional institutions:

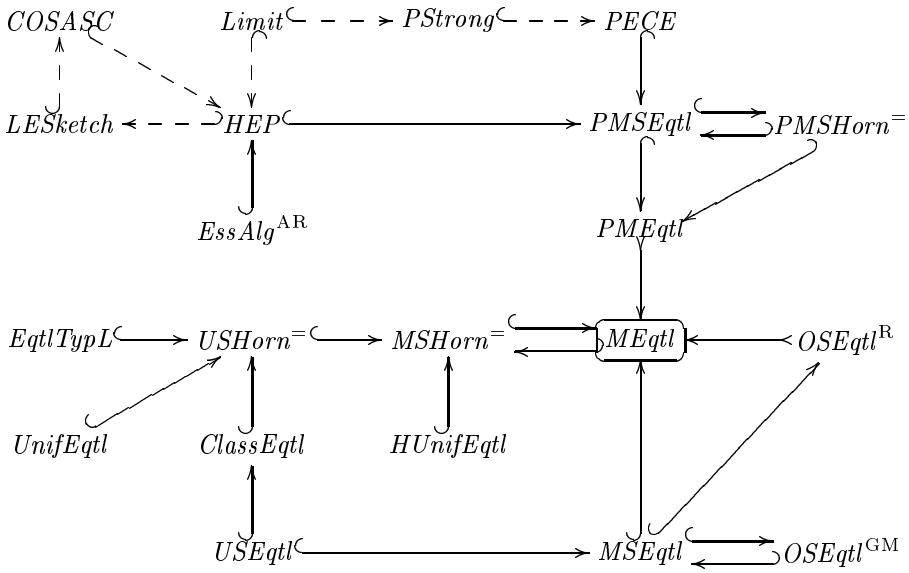
- *LESketch* is the institution of models specifiable as limit-preserving functors into **Set** for left-exact sketches [2].
- *Limit* is the institution of models for limit theories in the sense of Coste [10].
- *PStrong* is the institution of partial many-sorted equational logic with *strong* equations [5].
- *PECE* is the sublogic of *PMSEqtl* where all equations in a clause's conditions are "definedness" existence equations of the form $t = t$.
- *COSASC* is the institution of conditional order-sorted algebras with sort constraints in the sense of [14, 25].

In addition, we consider the institution $PMSHorn^{=}$ of partial many-sorted Horn logic with equality, where the models are partial many-sorted algebras together with predicates, and the sentences are Horn clauses whose atomic formulas are either existence equations or predicates, with $M, a \models P(t_1, \ldots, t_n)$ meaning that $\overline{a}(t_i)$ is defined for $1 \leq i \leq n$, and $(\overline{a}(t_1), \ldots, \overline{a}(t_n)) \in P_M$. In a way entirely analogous to the embedding $MSHorn^{=} \hookrightarrow MEqtl$ studied in Section 3, we also have an embedding[14] $PMSHorn^{=} \hookrightarrow PMEqtl$. Furthermore, Mossakowski has also constructed mutual embeddings between *PMSEqtl* and an institution *RPCEL* essentially equivalent to $PMSHorn^{=}$.

The point is that, following such a web, all these institutions can be embedded, or weakly embedded, in *PMEqtl*, which in turn we have shown can be mapped by an extension map into *MEqtl*. We can summarize the entire situation, including our previous discussion of total equational specification formalisms in Section 8, in the following diagram, which justifies our claim that membership equational logic is indeed a suitable and simple logical framework for equational

[14] To get an embedding in the other direction, one should restrict the signature morphisms in $\mathbf{Sign}_p$ by requiring that they preserve tops of connected components.

specification.



16 Concluding Remarks

I discuss below a number of future developments that seem particularly useful to exploit the results of this work in practice.

A matter that can be improved is the somewhat verbose translation of theories in the extension map $(\Psi, \gamma, (\_)^\circ) : \mathit{PMEqtl} \rightarrowtail \mathit{MEqtl}$, due to the replacement of atomic formulas by their corresponding envelopes. In practice many theories $T \in \mathbf{Th}_p$ are *strict*, in the precise sense that $\Psi(T)$ defines the same class of algebras as the theory obtained by the straightforward translation that maps a sentence $(\forall \tilde{X})\ at \Leftarrow at_1 \wedge \ldots \wedge at_n$ to the sentence $(\forall X)\ at \Leftarrow at_1 \wedge \ldots \wedge at_n \wedge \tilde{X}$. Sufficient conditions and algorithms to guarantee strictness will be discussed in future work.

The summary of logics that can be represented in *MEqtl* given in Section 15 suggests a natural path for mapping the Horn fragment *CASLHorn* of the CASL logic into membership equational logic. Theories in this fragment can be embedded very naturally into partial many-sorted Horn logic with equality $\mathit{PMSHorn}^=$. We then get the desired representation as the composition

$$\mathit{CASLHorn} \hookrightarrow \mathit{PMSHorn}^= \hookrightarrow \mathit{PMEqtl} \rightarrowtail \mathit{MEqtl}\ .$$

A somewhat less general subset of *CASLHorn* with subsorts understood as inclusions instead of as coercions, and with no extra predicates would of course have a straightforward embedding in *PMEqtl*, and would yield a very simple mapping into *MEqtl* if strictness checks of the kind mentioned above are used.

Besides having a logic with good properties, one is also interested in having theorem-proving techniques to verify properties of specifications in such a logic, including inductive properties of specifications with an initial semantics and of parameterized specifications. In this regard the prospects look quite promising. In joint work with Adel Bouhoula and Jean-Pierre Jouannaud [4, 3], we are developing a systematic study of such theorem-proving techniques, including Knuth-Bendix completion, tests for protecting and sufficiently complete theory extensions, and inductive theorem-proving techniques that extend to membership equational logic the rich body of results developed for many-sorted and order-sorted equational logics. Using the reflective capabilities of Maude we are beginning to build at SRI some theorem-proving tools written in Maude that implement some of these techniques and that will be used in the Cafe language and also in Maude itself.

The reflective character of rewriting logic and Maude [8] can also be very useful for implementing the actual representation map $(\Phi, \alpha, \beta) : \mathcal{L} \to \mathit{MEqtl}$ of a logic $\mathcal{L}$ into *MEqtl*. Although such a map is clearly a metalevel entity, it can be brought down to the object level by observing that, by reflection, there is an equationally defined data type $\mathtt{ADT}_{\mathit{MEqtl}}$ of theories in *MEqtl*, and that we can then implement the map of theories inside *MEqtl* by defining another data type $\mathtt{ADT}_{\mathcal{L}}$ for theories in $\mathcal{L}$ and an equationally defined function $\overline{\Phi} : \mathtt{ADT}_{\mathcal{L}} \to \mathtt{ADT}_{\mathit{MEqtl}}$ representing Φ. This technique, illustrated in [22], together with the reflective features of Maude, that can compile and execute the theories in $\mathtt{ADT}_{\mathit{MEqtl}}$, can then be used to translate a logic into *MEqtl* and to execute the translated theory.

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